On the Decidability of the Reachability Problem for Planar Differential Inclusions

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Overview of the presentation

- Introduction

- Simple Planar Differential Inclusion System (SPDI)

  - Reachability Problem

    - Difficulties

    - Our solution

  - Reachability Al orithm (Example)

- Conclusions

Introduction

- Reachability for Hybrid Systems
  - Undecidable in most cases
  - Approximation methods
    - Polyhedral approximations
    - Ellipsoidal approximations
    - Level-set approximations
  - Exact decision methods for sub-classes
    - Finite bisimulation (TA, Initialized Rectan ular Automata, etc.)
    - Quantifier elimination
    - "Geometric" methods (only on the plane)

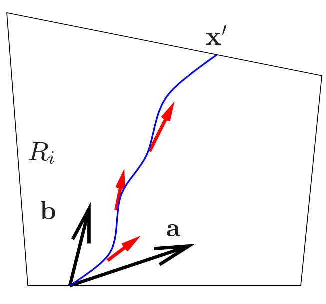
Introduction (Cont.)

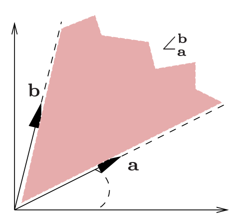
- “Geometric” methods - Ideas
  - Based on: Topolo y + Dynamical Systems
  - Require: Explicit solutions of Differential Equations
  - Take advanta e of:
    * Topolo y of the plane
    * Explicit formula for Successors
    * Acceleration of cycles

SPDI: Simple Planar Differential Inclusion System

- SPDI:
  - A partition of the plane into convex poly onal re ions
  - A constant differential inclusion for each re ion

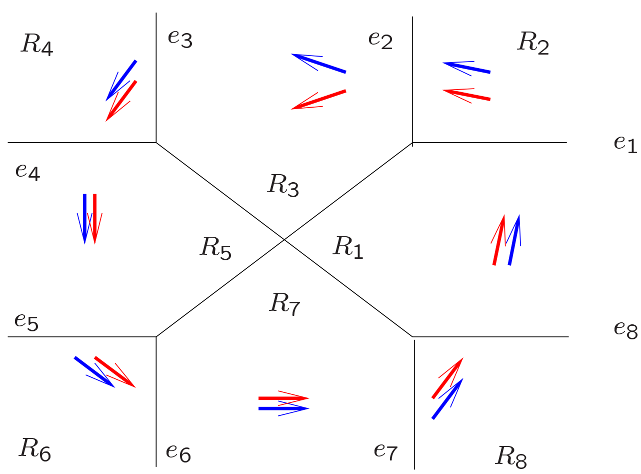
$$\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}} \text{ if } \mathbf{x} \in R_i$$





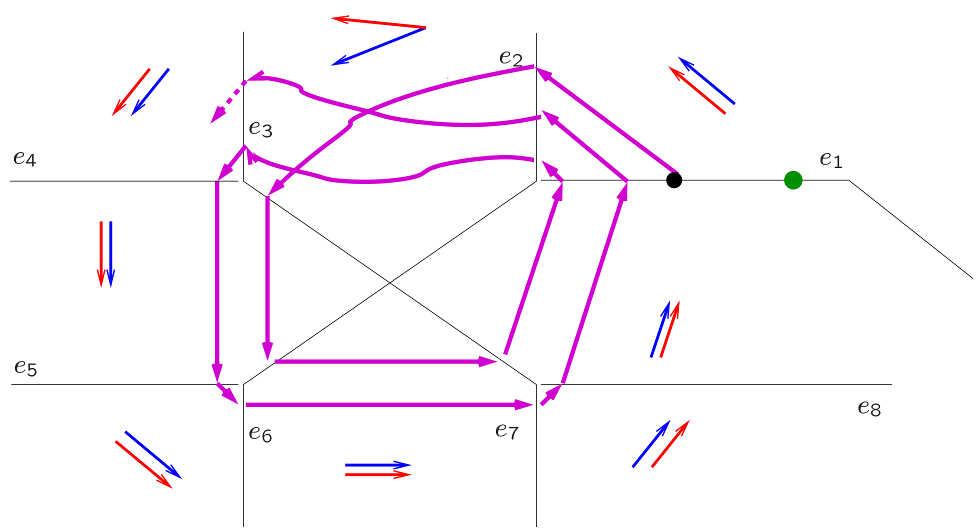
SPDI: Simple Planar Differential Inclusion System

Example: Swimmer tryin to escape from a whirlpool



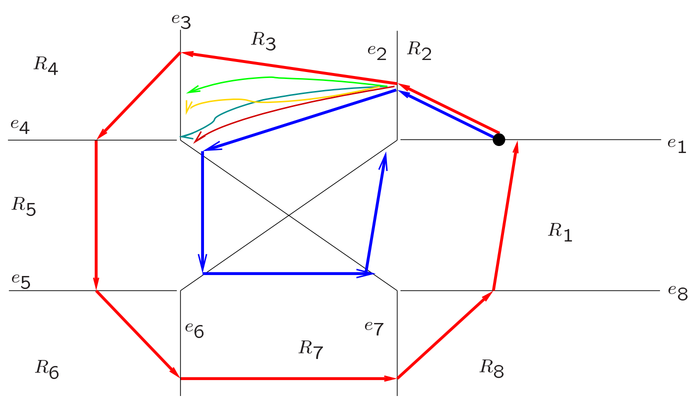
Reachability Problem

Given $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$, is there a trajectory ξ and $t \geq 0$ such that $\xi(0) = \mathbf{x}$ and $\xi(t) = \mathbf{x}'$?



Reachability Difficulties

Infinitely many trajectories (even locally)

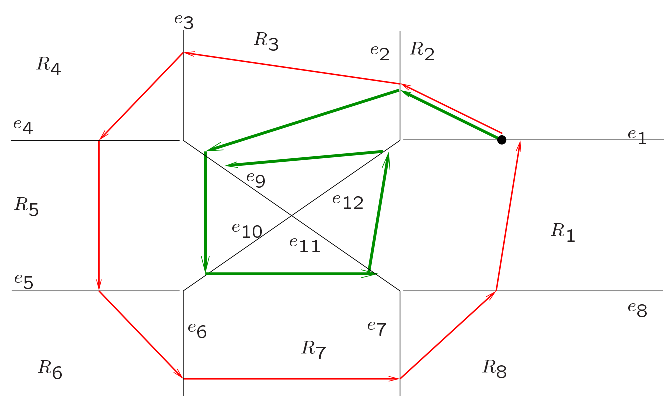


Reachability Difficulties

Infinitely many qualitative behaviors:

$\xi_1 = e_1 e_2 \; e_9 e_{10} e_{11} e_{12} \; e_9 \cdots e_{12} \cdots$

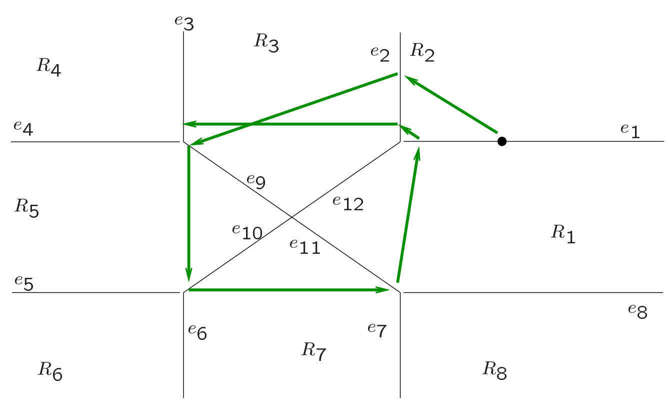
$\xi_2 = e_1 e_2 \cdots e_8 e_1 \cdots$



Reachability Difficulties

Self-crossin s:

$$\xi_3 = e_1 \; e_2 e_9 e_{10} e_{11} e_1 \; e_2 e_3 \cdots$$



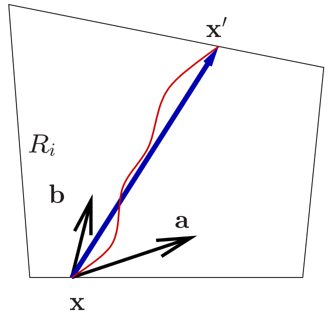
Reachability (Solution)

Our approach:

- Simplification of trajectories
- Enumeration of qualitative behavior of trajectories
- Analysis of each qualitative behavior (computin successors)

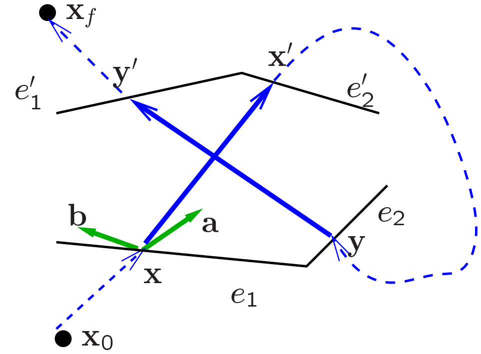
Simplification of Trajectories

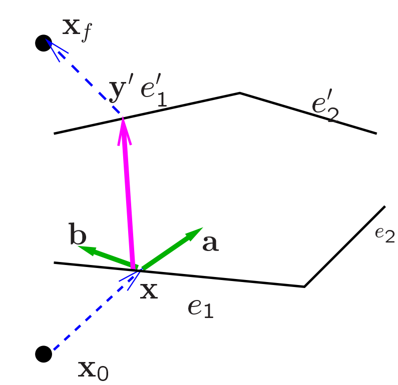
Strai htenin :



Simplification of Trajectories (Cont.)

Removin self-crossin s:





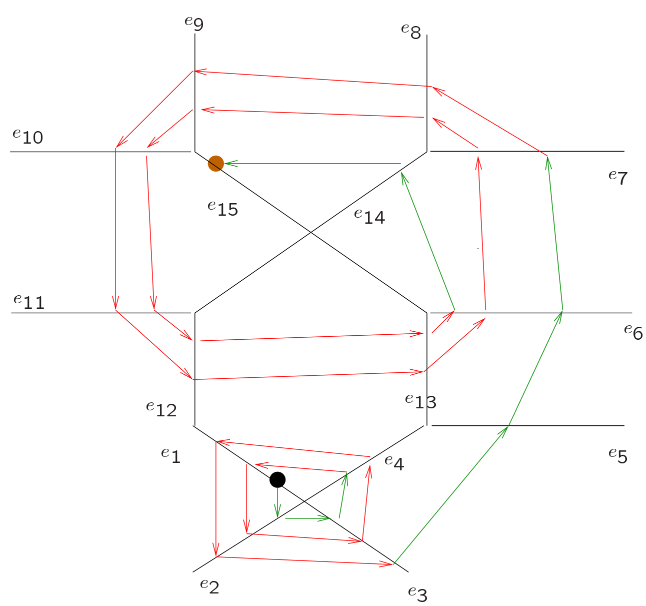
Qualitative Behavior

(of Simplified Trajectories)

- Classification Theorem: Any ed e si na- ture. σ belon s to a type

$$type(\sigma) = r_1(s_1)^* r_2(s_2)^* \dots r_n(s_n)^* r_{n+1}$$

- Example:



with $\sigma =$
$e_1 e_2 e_3\ (e_4 e_1 e_2 e_3)^2 e_5 e_6\ (e_7 \cdots e_{13} e_6)^2\ e_{14} e_{15}$

Qualitative Behavior

(of Simplified Trajectories)

- Properties:
  - r_i is a seq. of pairwise different edges;
  - s_i is a simple cycle;
  - r_i and r_j are disjoint
  - s_i and s_j are different

- **Prop.** The set of type of signatures is finite

Preliminaries - for computing Successors

- Affine function (AF):
  $f(x) = ax + b$ with $a > 0$

- Affine multi-valued function (AMVF):
  $\tilde{F}(x) = \langle f_1(x), f_2(x) \rangle$

- Truncated affine multi-valued function (TAMVF): $F(x) = \tilde{F}(x) \cap \langle L, U \rangle$

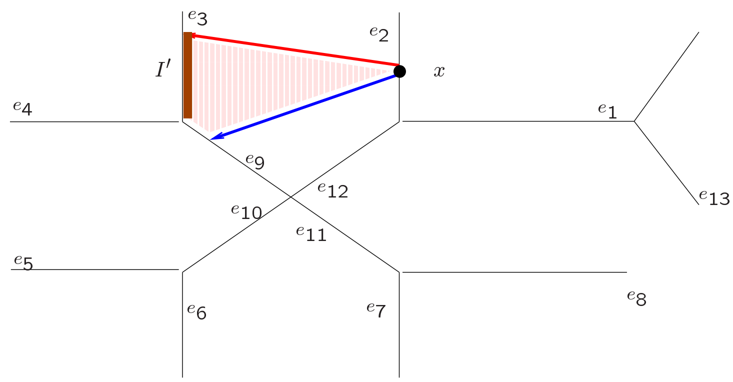
Lemma: AF, AMVF and TAMVF are closed under composition.

Lemma: Fixpoint equations $F(I) = I$ can be explicitely solved (without iteratin)*.

*Non trivial for T MVF.

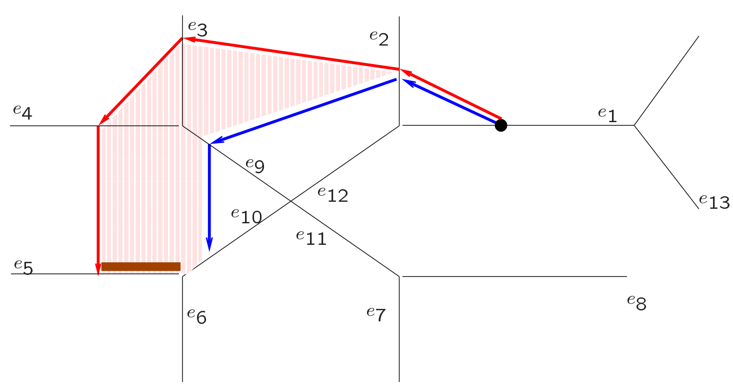
Successors (by σ)

- One step ($\sigma = e_2 e_3$)



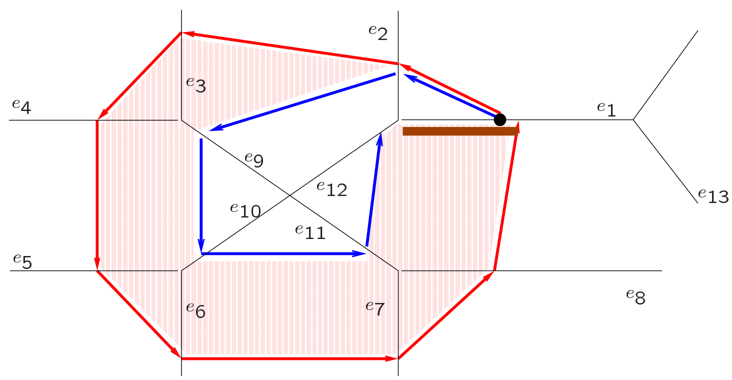
$$I' = \mathsf{Succ}_{e_1 e_2}(x) = [f_{\mathbf{b}}(x), f_{\mathbf{a}}(x)] = F(x)$$

- Several steps ($\sigma = e_1 e_2 e_3 e_4 e_5$)

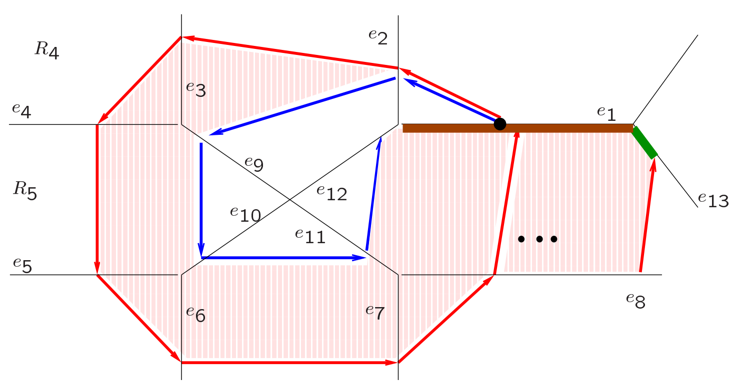


Successors (by σ)

- One cycle ($\sigma = s = e_1 e_2 \cdots e_8 e_1$)



- One cycle iterated: *solution of fixpoint equation (acceleration)* ($\sigma = (s)^* e_{13}$)



Reachability Al orithm

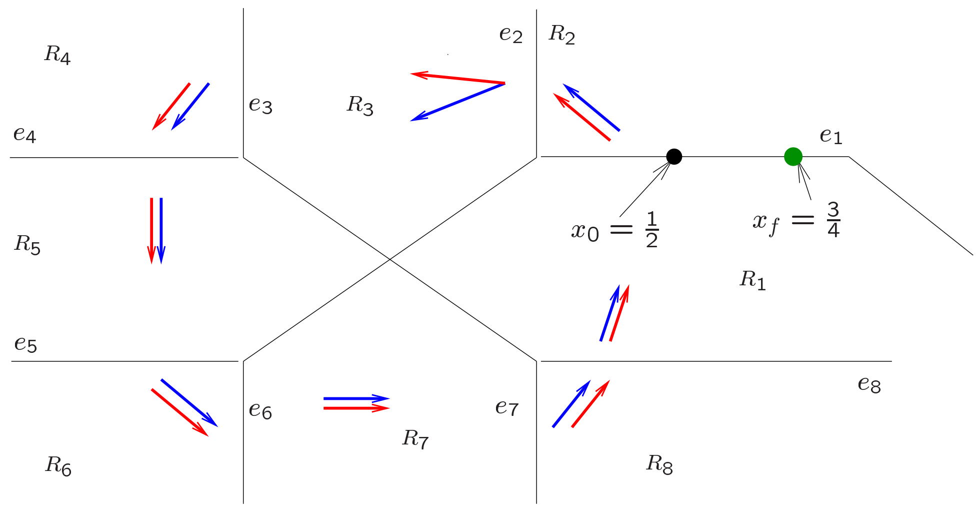
- We have that

$$x_f \in Reach(x_0)$$
$$\text{iff}$$
$$\text{for some } \sigma,\ \ x_f \in Succ_\sigma(x_0)$$

- Al orithm: Let $x_0 \in e_0$ and $x_f \in e_f$.
  For each type $type(\sigma) = r_1(s_1)^* \cdots (s_n)^* r_{n+1}$, with $e_0 = first(r_1)$ and $e_f = last(r_{n+1})$ check whether

$$x_f \in Succ_\sigma(x_0)$$

Reachability Al orithm (Example)



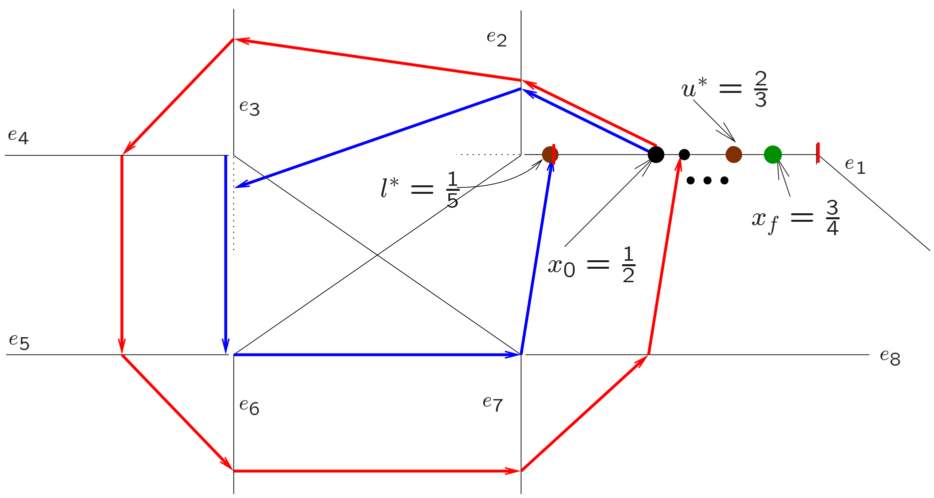
- Type of si nature: $\sigma = (e_1 \cdots e_8)^*$

- Successor for the loop $s = e_1 \ldots e_8$:

$$\mathrm{Succ}_{e_1 \ldots e_8}(l, u) = [\frac{l}{2} - \frac{1}{10}, \frac{u}{2} + \frac{1}{3}] \cap (\frac{1}{5}, 1)$$

ili l ri l

- ix oint e uation: ucc$_{e_1 \ldots e_8}$ $*$ $=$ $*$
- olution: $* = [l^*, u^*] = [\frac{1}{5}, \frac{2}{3}]$
- ence: ucc$_{e_1 \ldots e_8}$ x_0 $\subseteq [\frac{1}{5}, \frac{2}{3}]$



onclusion: x $/$ $*$ $\frac{3}{4}$ $/$ $[\frac{1}{5}, \frac{2}{3}]$.
ence, $\mathbf{x}$ $/$ eac $\mathbf{x}_0$

N **:** t e solution as foun it out iteratin *accelerati n* .

n lusi ns i n s

- eac a ilit is eci a le for I
- irst a lication of “ eo etric” et o s to lanar non- eter inistic s ste s
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n lusi ns Fu ur W rks

- o lexit anal sis
- xtensions:
  - ierarc ical I’s
  - 2- i surfaces ex.: torus
  - in in li it c cles
- lications:
  - roxi ation of non-linear i erential e uations

“ s” l s

1. e c cle is not a an one an of t e t o trajectories: $\leq l^* \leq u^* \leq$.

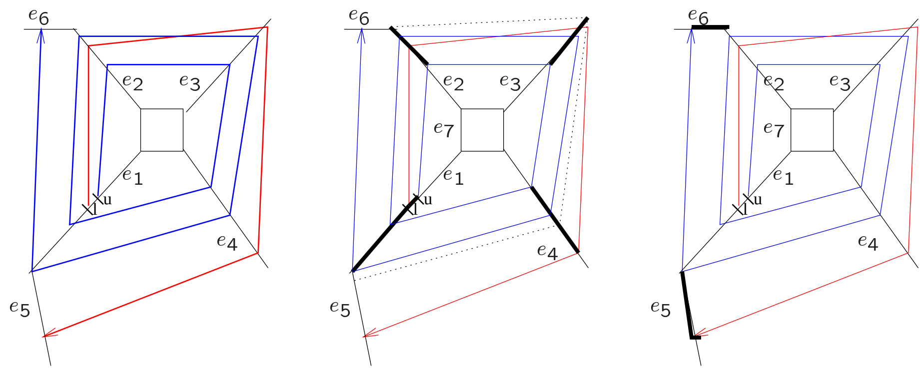
2. **I** e ri t trajector exits t e c cle t rou t e left conse uentl t e left one also exits : $u^* < \vee l^*$.

3. **XI - H** ot trajectories exit t e c cle t e left one t rou t e left an t e ri t one t rou t e ri t : $l^* < \wedge u^*$.

4. **XI - F** e left ost trajector exits t e c cle ut not t e ot er: $l^* < \leq u^* \leq$. ual: XI - I .

“ s” l s

- ie



(a) (b) (c)

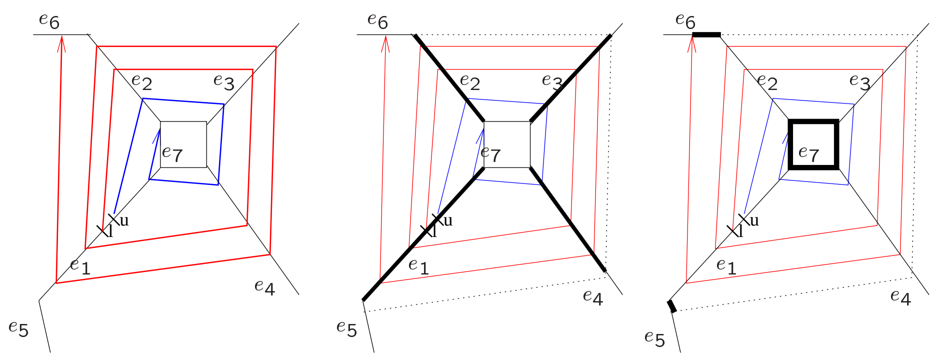
  - a ot trajectories leave t e c cle e_1, e_2, e_3, e_4 * t rou t e left, ereas t e ri t one ten s to t e li it u^*;

  - eac a le oints on t e c cle in ol ;

  - c ossi le continuation after leavin t e c cle in ol .

“ s” l s

- xit ot



(a) (b) (c)

  - a ot trajectories leave t e c cle e_1, e_2, e_3, e_4 *;
  - eac a le oints on t e c cle in ol ;
  - c ossi le continuation after leavin t e c cle in ol .

“ s” l s

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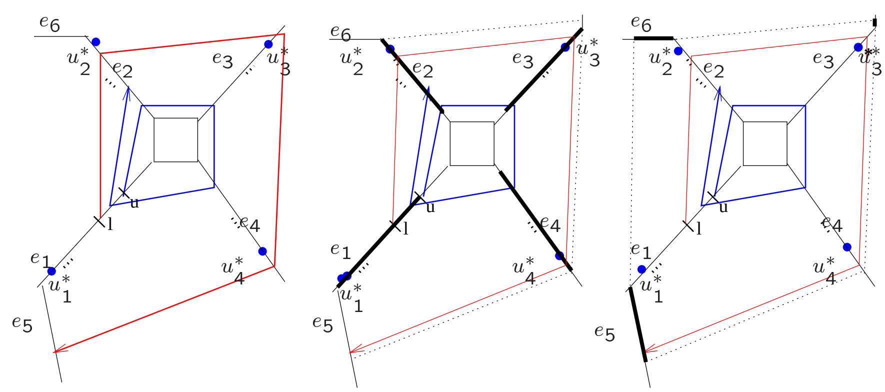
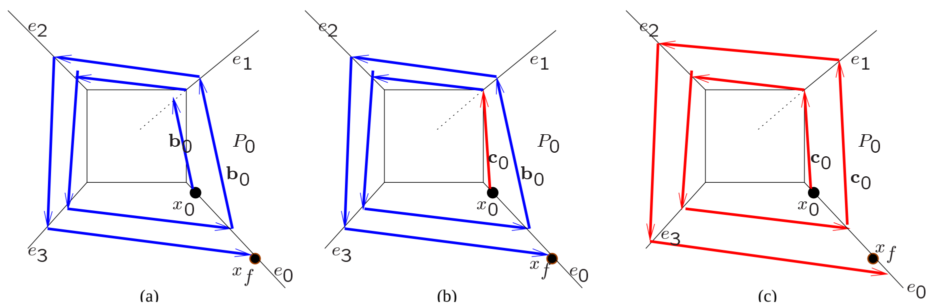


(a) (b) (c)

  - a e left trajector leave t e c cle e_1, e_2, e_3, e_4 * t rou t e left;
  - eac a le oints on t e c cle in ol ;
  - c ossi le continuation after leavin t e c cle in ol .

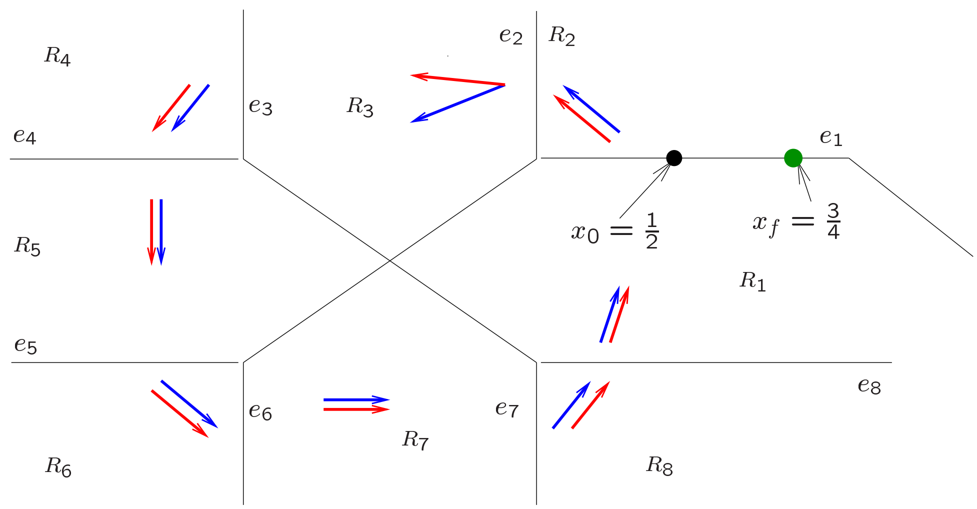
r ri n sis r l

- eci a ilit of reac a ilit for I oes not i l eci a ilit for t e ara etric case.



  - a x is reac a le fro x_0 follo in t e left ost truncate trajector ;
  - ara eter ust e “at least” $\mathbf{c}_0$;
  - c a in $\mathbf{c}_0$ a es x unreac a le.

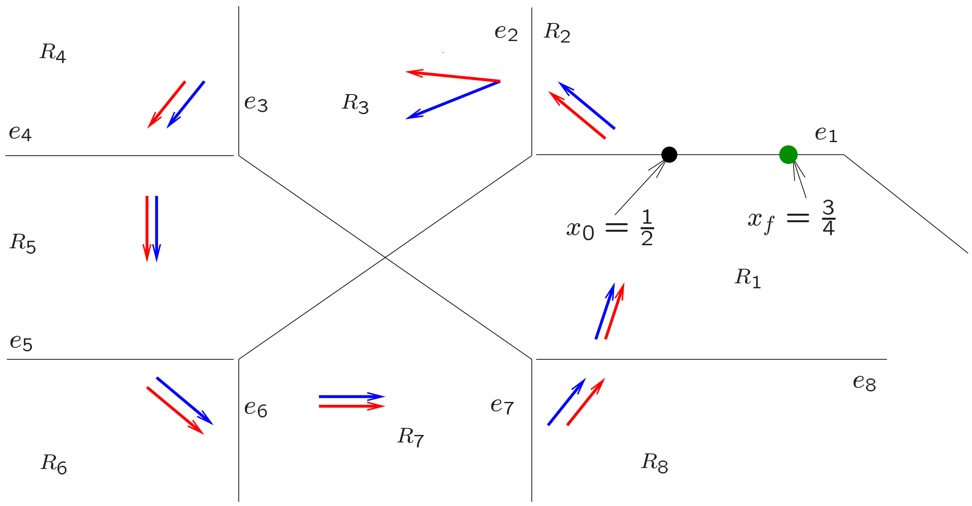
ili l ri l



- na ics:
  - $\mathbf{a}_1 = {}_1 = (1, 5)$,
  - $\mathbf{a}_2 = {}_2 = (-1, \frac{1}{2})$,
  - $\mathbf{a}_3 = (-1, \frac{11}{60})$ and ${}_3 = (-1, -\frac{1}{4})$,
  - $\mathbf{a}_4 = {}_4 = (-1, -1)$,
  - $\mathbf{a}_5 = {}_5 = (0, -1)$,
  - $\mathbf{a}_6 = {}_6 = (1, -1)$,
  - $\mathbf{a}_7 = {}_7 = (1, 0)$,
  - $\mathbf{a}_8 = {}_8 = (1, 1)$.

- ype of signature: $\sigma = (e_1 \cdots e_8)^*$

ili l ri l



- uccessor functions are:

  $$\text{ucc}_{e_1e_2} \; x \;\; = \left[\frac{x}{2}, \frac{x}{2}\right] \qquad ,1$$
  $$\text{ucc}_{e_2e_3} \; x \;\; = \left[x - \frac{3}{10}, x \quad \frac{2}{15}\right] \qquad ,1$$
  $$\text{ucc}_{e_ie_{i+1}} \; x \;\; = [x, x] \qquad ,1 \;\; , \text{ for all } i \quad 3..7$$
  $$\text{ucc}_{e_8e_1} \; x \;\; = \left[x \quad \frac{1}{5}, x \quad \frac{1}{5}\right] \qquad ,1$$

- uccessor for t e loo $\;\; s = e_1 \ldots e_8$:

  $$\text{ucc}_{e_1\ldots e_8} \; l, u \;\; = [\tfrac{l}{2} - \tfrac{1}{10}, \tfrac{u}{2} \quad \tfrac{1}{3}] \quad \tfrac{1}{5}, 1$$