Computing Invariance Kernels of Polygonal Hybrid Systems*

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1 Introduction

A *hybrid system* [ACH+95] is a system where both continuous and discrete behaviors interact with each other. A typical example is given by a discrete program that interacts with a continuous physical environment. One of the main research areas in hybrid systems is reachability analysis. Most of the proved decidability results are based on the existence of a finite and computable partition of the state space into classes of states which are equivalent with respect to reachability. For some particular classes of two-dimensional dynamical systems a *geometrical* method, which relies on the analysis of topological properties of the plane, has been developed. This approach has been proposed in [MP93]. There, it is shown that the reachability problem for two-dimensional systems with piece-wise constant derivatives (PCD) is decidable. This result has been extended in [CV96] for planar piece-wise Hamiltonian systems and in [ASY01] for polygonal hybrid systems, a class of nondeterministic systems that correspond to piecewise constant differential inclusions on the plane (SPDIs [Sch02]). In [AMP95] it has been shown that the reachability problem for PCD is undecidable for dimensions higher than two.

Another important issue in the analysis of a (hybrid) dynamical system is the study of its qualitative behavior, namely the construction of its *phase portrait*. Some typical questions on this sense are "does every trajectory except the equilibrium point in the origin converges to a limit cycle which is the unit circle?", or "what is the biggest set such that any point on it is reachable from any other point on the set?". There have been very few results on the qualitative properties of trajectories of hybrid systems [ASY02,Aub01,DV95,KV96,KdB01,MS00,SJSL00]. In particular, the question of defining and constructing phase portraits of hybrid systems has not been directly addressed except in [MS00], where phase portraits of deterministic systems with piecewise constant derivatives are explored and in [ASY02] where viability and controllability kernels for polygonal differential inclusion systems have been computed.

In this work we discuss how to compute another important object of phase portraits of SPDIs, namely the *invariance kernel*. In general, an *invariant set* is a set of points such that for any point in the set, every trajectory starting in such point remains in the set forever and the *invariance kernel* is the largest of such sets. We show that, for SPDIs, this kernel is a non-convex polygon and we give a non-iterative algorithm for computing the coordinates of its vertices and edges. Clearly, the invariance kernel provides useful insight about the behavior of the SPDI around simple cycles.

In the next section we present, informally, the definition of SPDI and we discuss how to compute its invariance kernel for simple cycles. In the last section we present some concluding remarks.

2 Invariance Kernels of SPDIs

A *polygonal differential inclusion system* (SPDI) consists of a partition of the plane into convex polygonal regions, together with a differential inclusion (given by two vectors) associated with each

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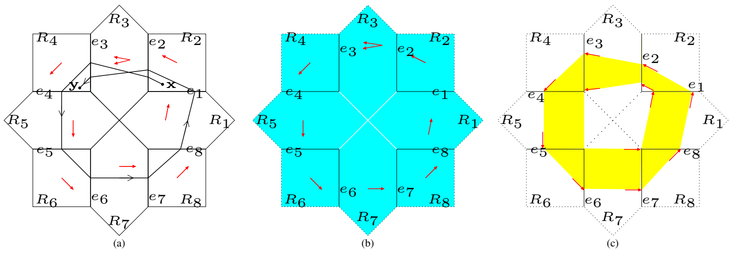

Fig. 1. (a) An SPDI and its trajectory segment; (b) The set K_σ; (c) Invariance Kernel

region. An example of a trajectory of such systems is depicted in Figure 1-(a). The main problem in analysing SPDIs is their non-deterministic behaviour due to the fact that the dynamics in each region is given by a differential inclusion. Clearly, even for simple systems like the one shown in the figure, there could be infinite trajectories between any two points.

One of the interesting results (useful for our purpose in this work) shown in [ASY01] is the classification of simple cycles in five different types. Indeed, given a simple cycle, it can be of one of the following forms: STAY, the cycle is not abandoned neither by the leftmost nor the rightmost trajectory; DIE, the rightmost trajectory exits the cycle through the left (consequently the leftmost one also exits) or the leftmost trajectory exits the cycle through the right (consequently the rightmost one also exits); EXIT-BOTH, the leftmost trajectory exits the cycle through the left and the rightmost one through the right; EXIT-LEFT, the leftmost trajectory exits the cycle (through the left) but the rightmost one stays inside; EXIT-RIGHT, the rightmost trajectory exits the cycle (through the right) but the leftmost one stays inside. The above classification on simple cycles gives an intuitive idea about where to "find" the invariance kernels: only STAY cycles can have a non-empty invariance kernel, since in any other type of cycle the trajectories would eventually leave the cycle.

In order to compute the invariance kernel we need to define the notion of one-dimensional discrete-time *edge-to-edge successors*. Given two edges e and e' of a given region P of the SPDI, and an interval $I \subseteq e$, $\mathsf{Succ}_{e,e'}(I)$ is the set of all points in e' reachable from some point in I by a trajectory segment $\xi : [0,t] \to \mathbb{R}^2$ in P. Essentially, these operators are one-dimensional multi-valued affine functions from intervals to intervals: It can be shown that $\mathsf{Succ}_{e,e'}([l,u]) = [a_1 l + b_1, a_2 u + b_2]$, with $a1, a_2 > 0$.

We can define in a similar way the *edge-to-edge predecessors* of an interval: For $I \subseteq e'$, $\mathsf{Pre}_{e,e'}(I)$ is the set of points in e that can reach a point in I by a trajectory segment in P. We define the $\forall$-predecessor $\widetilde{\mathsf{Pre}}_{e,e'}(I)$ as the set of points in e such that *any* successor of such points are in I by a trajectory segment in P.

Successors and predecessors can be extended to sequences of edges $\sigma = e_1, \ldots e_n$, and since they are closed under compositions, the obtained function is again a multi-valued affine function. Notice that in particular e_1 can be equal to e_n and in that case we call σ a *cyclic signature*. Let us denote by J the co-domain of Succ_σ.

Let $K \subset \mathbb{R}^2$ and K_σ the open (non-convex) polygon obtained considering the interior of all the regions that contains the edges in σ union the edges in σ. In Figure 1-(b) we have K_σ for $\sigma = e_1, \ldots e_8$. The invariance kernel $\mathsf{Inv}(K_\sigma)$ for the continuous-time system can be now found by propagating $\widetilde{\mathsf{Pre}}(J)$ from e_1 using the following operator. The *σ-extended $\forall$-predecessor* of I, $\overline{\widetilde{\mathsf{Pre}}}_\sigma(I)$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ for which any trajectory segment ξ starting in $\mathbf{x}$, reaches some point in I.

We have then the main result of this paper:

Theorem 1. *If σ is STAY then* $\mathsf{Inv}(K_\sigma) = \overline{\widetilde{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$, *otherwise* $\mathsf{Inv}(K_\sigma) = \emptyset$. □

The invariance kernel corresponding to the SPDI of Figure 1-(a) is shown in Figure 1-(c).

3 Conclusion

We have presented in this extended abstract the outline of an automatic procedure to obtain an important object of the phase portrait of simple planar differential inclusions (SPDIs [Sch02]), namely invariance kernels. The main feature is that we can compute such kernels with non-iterative algorithm and in an exact way, which is not the case in general for many classes of hybrid systems. Formal definitions and proofs can be found in [Sch03].

Besides the inherent importance of invariance kernels as for eventual applications of SPDIs for approximating non-linear differential equations, they play a key role for proving termination of a breadth-first search reachability algorithm for SPDIs ([**?**]).

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