Towards computing phase portraits of polygonal differential inclusions ⋆

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Abstract. Polygonal hybrid systems are a subclass of planar hybrid automata which can be represented by piecewise constant differential inclusions. Here, we study the problem of defining and constructing the phase portrait of such systems. We identify various important elements of it, such as viability and controllability kernels, and propose an algorithm for computing them all. The algorithm is based on a geometric analysis of trajectories.

1 Introduction

Given a (hybrid) dynamical system one can ask whether a point (or set) is reachable from another, or one can ask for a full qualitative picture of the system (say, its phase portrait). An answer to the second question provides very useful information about the behavior of the system such as "every trajectory except the equilibrium point in the origin converges to a limit cycle which is the unit circle". The reachability question has been an important and extensively studied research problem in the hybrid systems community. However, there have been very few results on the qualitative properties of trajectories of hybrid systems [1, 3, 5, 7–10]. In particular, the question of defining and constructing phase portraits of hybrid systems has not been directly addressed except in [9], where phase portraits of deterministic systems with piecewise constant derivatives are explored.

In this paper we study phase portraits of polygonal hybrid systems (or, SPDIs), a class of nondeterministic systems that correspond to piecewise constant differential inclusions on the plane (Fig. 1). It is not a priori clear what the phase portraits of such systems exactly are. To begin with, we concentrate on studying the qualitative behavior of sets of trajectories having the same cyclic pattern. In [1], we have given a classification of cyclic behaviors. Here, we rely on this information to more deeply study the qualitative behavior of the system. In particular, we are able to compute the *viability* kernel [2, 4] of the cycle, that is, the set of points which can keep rotating in the cycle forever. We show that this kernel is a non-convex polygon (often with a hole in the middle) and give a

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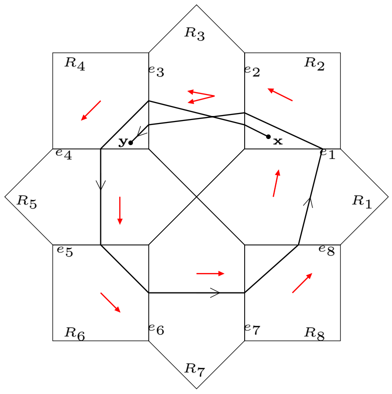

Fig. 1. An SPDI and its trajectory segment.

non-iterative algorithm for computing the coordinates of its vertices and edges. Clearly, the viability kernel provides useful insight about the behavior of the SPDI around the cycle. Furthermore, we are also (and even more) interested in the limit behaviors. We introduce a notion of *controllability* kernel, a cyclic polygonal stripe within which a trajectory can reach any point from any point. We show how to compute it and argue that this is a good analog of the notion of limit cycle. Indeed, we prove that the distance between any infinite trajectory performing forever the same cyclic pattern and the controllability kernel always converges to zero.

In section 4 we show that any simple (without self-crossings) infinite trajectory converges to one of those "limit cycles" (controllability kernels). We conclude that controllability kernels are important elements of the phase portrait of an SPDI yielding an analog of Poincaré-Bendixson theorem for simple trajectories. We apply all these results to compute (elements of) the phase portrait by enumerating all the feasible cyclic patterns and computing its viability and controllability kernels. We also discuss difficulties related to self-crossing trajectories, which can, for example, randomly walk in two adjacent controllability kernels.

2 Preliminaries

2.1 Truncated affine multivalued functions

A (positive) *affine* function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x) = ax + b$ with $a > 0$. An *affine multivalued* function $F : \mathbb{R} \to 2^{\mathbb{R}}$, denoted $F = \langle f_l, f_u \rangle$, is defined by $F(x) = \langle f_l(x), f_u(x) \rangle$ where f_l and f_u are affine and $\langle \cdot, \cdot \rangle$ denotes an interval. For notational convenience, we do not make explicit whether intervals are open, closed, left-open or right-open, unless required for comprehension. For an interval $I = \langle l, u \rangle$ we have that $F(\langle l, u \rangle) = \langle f_l(l), f_u(u) \rangle$. The inverse of F is defined by

$F^{-1}(x) = \{y \mid x \in F(y)\}$. It is not difficult to show that $F^{-1} = \langle f_u^{-1}, f_l^{-1} \rangle$. These classes of functions are closed under composition.

A *truncated affine multivalued* function (TAMF) $\mathcal{F} : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by an affine multivalued function F and intervals $S \subseteq \mathbb{R}^+$ and $J \subseteq \mathbb{R}^+$ as follows: $\mathcal{F}(x) = F(x) \cap J$ if $x \in S$, otherwise $\mathcal{F}(x) = \emptyset$. For convenience we write $\mathcal{F}(x) = F(\{x\} \cap S) \cap J$. For an interval I, $\mathcal{F}(I) = F(I \cap S) \cap J$ and $\mathcal{F}^{-1}(I) = F^{-1}(I \cap J) \cap S$. We say that $\mathcal{F}$ is *normalized* if $S = \mathsf{Dom}\mathcal{F} = \{x \mid F(x) \cap J \neq \emptyset\}$ (thus, $S \subseteq F^{-1}(J)$) and $J = \mathsf{Im}\mathcal{F} = \mathcal{F}(S)$.

The following theorem states that TAMFs are closed under composition [1].

Theorem 1. The composition of two TAMFs $\mathcal{F}_1(I) = F_1(I \cap S_1) \cap J_1$ and $\mathcal{F}_2(I) = F_2(I \cap S_2) \cap J_2$, is the TAMF $(\mathcal{F}_2 \circ \mathcal{F}_1)(I) = \mathcal{F}(I) = F(I \cap S) \cap J$, where $F = F_2 \circ F_1$, $S = S_1 \cap F_1^{-1}(J_1 \cap S_2)$ and $J = J_2 \cap F_2(J_1 \cap S_2)$.

2.2 SPDI

An *angle* $\angle_{\mathbf{a}}^{\mathbf{b}}$ on the plane, defined by two non-zero vectors $\mathbf{a}, \mathbf{b}$ is the set of all positive linear combinations $\mathbf{x} = \alpha\ \mathbf{a} + \beta\ \mathbf{b}$, with $\alpha, \beta \geq 0$, and $\alpha + \beta > 0$. We can always assume that $\mathbf{b}$ is situated in the counter-clockwise direction from $\mathbf{a}$.

A *simple planar differential inclusion* (SPDI) is defined by giving a finite partition $\mathbb{P}$ of the plane into convex polygonal sets, and associating with each $P \in \mathbb{P}$ a couple of vectors $\mathbf{a}_P$ and $\mathbf{b}_P$. Let $\phi(P) = \angle_{\mathbf{a}_P}^{\mathbf{b}_P}$. The SPDI is $\dot{\mathbf{x}} \in \phi(P)$ for $\mathbf{x} \in P$.

Let $E(P)$ be the set of edges of P. We say that e is an *entry* of P if for all $\mathbf{x} \in e$ and for all $\mathbf{c} \in \phi(P)$, $\mathbf{x} + \mathbf{c}\epsilon \in P$ for some $\epsilon > 0$. We say that e is an *exit* of P if the same condition holds for some $\epsilon < 0$. We denote by $in(P) \subseteq E(P)$ the set of all entries of P and by $out(P) \subseteq E(P)$ the set of all exits of P.

Assumption 1 *All the edges in $E(P)$ are either entries or exits, that is, $E(P) = in(P) \cup out(P)$.*

Example 1. Consider the SPDI illustrated in Fig. 1. For each region R_i, $1 \leq i \leq 8$, there is a pair of vectors $(\mathbf{a}_i, \mathbf{b}_i)$, where: $\mathbf{a}_1 = \mathbf{b}_1 = (1, 5)$, $\mathbf{a}_2 = \mathbf{b}_2 = (-1, \frac{1}{2})$, $\mathbf{a}_3 = (-1, \frac{11}{60})$ and $\mathbf{b}_3 = (-1, -\frac{1}{4})$, $\mathbf{a}_4 = \mathbf{b}_4 = (-1, -1)$, $\mathbf{a}_5 = \mathbf{b}_5 = (0, -1)$, $\mathbf{a}_6 = \mathbf{b}_6 = (1, -1)$, $\mathbf{a}_7 = \mathbf{b}_7 = (1, 0)$, $\mathbf{a}_8 = \mathbf{b}_8 = (1, 1)$. ■

A *trajectory segment* of an SPDI is a continuous function $\xi : [0, T] \to \mathbb{R}^2$ which is smooth everywhere except in a discrete set of points, and such that for all $t \in [0, T]$, if $\xi(t) \in P$ and $\dot{\xi}(t)$ is defined then $\dot{\xi}(t) \in \phi(P)$. The *signature*, denoted $\mathsf{Sig}(\xi)$, is the ordered sequence of edges traversed by the trajectory segment, that is, $e_1, e_2, \ldots$, where $\xi(t_i) \in e_i$ and $t_i < t_{i+1}$. If $T = \infty$, a trajectory segment is called a *trajectory*.

Assumption 2 *We will only consider trajectories with infinite signatures.*

2.3 Successors and predecessors

Given an SPDI, we fix a one-dimensional coordinate system on each edge to represent points laying on edges [1]. For notational convenience, we indistinctly use letter e to denote the edge or its one-dimensional representation. Accordingly, we write $\mathbf{x} \in e$ or $x \in e$, to mean "point $\mathbf{x}$ in edge e with coordinate x in the one-dimensional coordinate system of e". The same convention is applied to sets of points of e represented as intervals (e.g., $\mathbf{x} \in I$ or $x \in I$, where $I \subseteq e$) and to trajectories (e.g., "ξ starting in x" or "ξ starting in $\mathbf{x}$").

Now, let $P \in \mathbb{P}$, $e \in in(P)$ and $e' \in out(P)$. For $I \subseteq e$, $\mathsf{Succ}_{e,e'}(I)$ is the set of all points in e' reachable from some point in I by a trajectory segment $\xi : [0,t] \to \mathbb{R}^2$ in P (i.e., $\xi(0) \in I \wedge \xi(t) \in e' \wedge \mathsf{Sig}(\xi) = ee'$). We have shown in [1] that $\mathsf{Succ}_{e,e'}$ is a TAMF[1].

Example 2. Let $e_1, \ldots, e_8$ be as in Fig. 1 and $I = [l, u]$. We assume a one-dimensional coordinate system such that $e_i = S_i = J_i = (0, 1)$. We have that:

$$F_{e_1e_2}(I) = \left[\frac{l}{2}, \frac{u}{2}\right] \quad F_{e_2e_3}(I) = \left[l - \frac{1}{4}, u + \frac{11}{60}\right]$$

$$F_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \quad F_{e_8e_1}(I) = \left[l + \frac{1}{5}, u + \frac{1}{5}\right]$$

with $\mathsf{Succ}_{e_ie_{i+1}}(I) = F_{e_ie_{i+1}}(I \cap S_i) \cap J_{i+1}$, for $1 \leq i \leq 7$, and $\mathsf{Succ}_{e_8e_1}(I) = F_{e_8e_1}(I \cap S_8) \cap J_1$. ∎

Given a sequence $w = e_1, e_2, \ldots, e_n$, Theorem 1 implies that the successor of I along w defined as $\mathsf{Succ}_w(I) = \mathsf{Succ}_{e_{n-1},e_n} \circ \ldots \circ \mathsf{Succ}_{e_1,e_2}(I)$ is a TAMF.

Example 3. Let $\sigma = e_1 \cdots e_8 e_1$. We have that $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$, where:

$$F(I) = \left[\frac{l}{2} - \frac{1}{20}, \frac{u}{2} + \frac{23}{60}\right] \tag{1}$$

$S = (0, 1)$ and $J = (\frac{1}{5}, 1)$ are computed using Theorem 1. ∎

For $I \subseteq e'$, $\mathsf{Pre}_{e,e'}(I)$ is the set of points in e that can reach a point in I by a trajectory segment in P. We have that[1]: $\mathsf{Pre}_{e,e'} = \mathsf{Succ}^{-1}_{e,e'}$ and $\mathsf{Pre}_\sigma = \mathsf{Succ}^{-1}_\sigma$.

Example 4. Let $\sigma = e_1 \ldots e_8 e_1$ be as in Fig. 1 and $I = [l, u]$. We have that $\mathsf{Pre}_{e_ie_{i+1}}(I) = F^{-1}_{e_ie_{i+1}}(I \cap J_{i+1}) \cap S_i$, for $1 \leq i \leq 7$, and $\mathsf{Pre}_{e_8e_1}(I) = F^{-1}_{e_8e_1}(I \cap J_1) \cap S_8$, where:

$$F^{-1}_{e_1e_2}(I) = [2l, 2u] \quad F^{-1}_{e_2e_3}(I) = \left[l - \frac{11}{60}, u + \frac{1}{4}\right]$$

$$F^{-1}_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \quad F^{-1}_{e_8e_1}(I) = \left[l - \frac{1}{5}, u - \frac{1}{5}\right]$$

Besides, $\mathsf{Pre}_\sigma(I) = F^{-1}(I \cap J) \cap S$, where $F^{-1}(I) = [2l - \frac{23}{30}, 2u + \frac{1}{10}]$. ∎

[1] In [1] we explain how to choose the positive direction on every edge in order to guarantee positive coefficients in the TAMF.

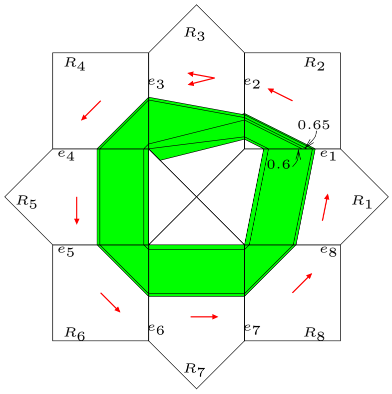

Fig. 2. Reachability analysis.

3 Qualitative analysis of simple edge-cycles

Let $\sigma = e_1 \cdots e_k e_1$ be a simple edge-cycle, i.e., $e_i \neq e_j$ for all $1 \leq i \neq j \leq k$. Let $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$ with $F = \langle f_l, f_u \rangle$ (we suppose that this representation is normalized). We denote by $\mathcal{D}_\sigma$ the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$.

Assumption 3 *None of the two functions f_l, f_u is the identity.*

Let l^* and u^* be the fixpoints[2] of f_l and f_u, respectively, and $S \cap J = \langle L, U \rangle$. We have shown in [1] that a simple cycle is of one of the following types:

STAY. The cycle is not abandoned neither by the leftmost nor the rightmost trajectory, that is, $L \leq l^* \leq u^* \leq U$.

DIE. The rightmost trajectory exits the cycle through the left (consequently the leftmost one also exits) or the leftmost trajectory exits the cycle through the right (consequently the rightmost one also exits), that is, $u^* < L \vee l^* > U$.

EXIT-BOTH. The leftmost trajectory exits the cycle through the left and the rightmost one through the right, that is, $l^* < L \wedge u^* > U$.

EXIT-LEFT. The leftmost trajectory exits the cycle (through the left) but the rightmost one stays inside, that is, $l^* < L \leq u^* \leq U$.

EXIT-RIGHT. The rightmost trajectory exits the cycle (through the right) but the leftmost one stays inside, that is, $L \leq l^* \leq U < u^*$.

Example 5. Let $\sigma = e_1 \cdots e_8 e_1$. We have that $S \cap J = \langle L, U \rangle = (\frac{1}{5}, 1)$. The fixpoints of Eq. (1) are such that $l^* = -\frac{1}{10} < \frac{1}{5} < u^* = \frac{23}{30} < 1$. Thus, σ is EXIT-LEFT. ■

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[2] Obviously, the fixpoint x^* is computed by solving a linear equation $f(x^*) = x^*$.

The classification above gives us some information about the qualitative behavior of trajectories. Any trajectory that enters a cycle of type DIE will eventually quit it after a finite number of turns. If the cycle is of type STAY, all trajectories that happen to enter it will keep turning inside it forever. In all other cases, some trajectories will turn for a while and then exit, and others will continue turning forever. This information is very useful for solving the reachability problem [1].

Example 6. Consider again the cycle $\sigma = e_1 \cdots e_8 e_1$. Fig. 2 shows part of the reach set of the interval $[0.6, 0.65] \subset e_1$. Notice that the leftmost trajectory exits the cycle in the third turn while the rightmost one shifts to the right and "converges to" the limit $u^* = \frac{23}{30}$. Clearly, no point in $[0.6, 0.65]$ will ever reach a point of e_1 smaller than $L = \frac{1}{5}$ or bigger than u^*. Fig. 2 has been automatically generated by the SPeeDi toolbox we have developed for reachability analysis of SPDIs based on the results of [1]. ■

The above result does not allow us to directly answer other questions about the behavior of the SPDI such as determine for a given point (or set of points) whether: (a) there exists (at least) one trajectory that remains in the cycle, and (b) it is possible to control the system to reach any other point. In order to do this, we need to further study the properties of the system around simple edge-cycles.

3.1 Viability kernel

Let $K \subset \mathbb{R}^2$. A trajectory ξ is *viable* in K if $\xi(t) \in K$ for all $t \geq 0$. K is a *viability domain* if for every $\mathbf{x} \in K$, there exists at least one trajectory ξ, with $\xi(0) = \mathbf{x}$, which is viable in K. The *viability kernel* of K, denoted $\mathsf{Viab}(K)$, is the largest viability domain contained in K[3]. The same concepts can be defined for $\mathcal{D}_\sigma$, by setting that a trajectory $x_0 x_1 \ldots$ of $\mathcal{D}_\sigma$ is viable in an interval $I \subseteq \mathbb{R}$, if $x_i \in I$ for all $i \geq 0$.

Theorem 2. *For $\mathcal{D}_\sigma$, if σ is not DIE then* $\mathsf{Viab}(e_1) = S$*, else* $\mathsf{Viab}(e_1) = \emptyset$.[4]

The viability kernel for the continuous-time system can be now found by propagating S from e_1 using the following operator.

For $I \subseteq e_1$ let us define $\overline{\mathsf{Pre}}_\sigma(I)$ as the set of all $\mathbf{x} \in \mathbb{R}^2$ for which there exists a trajectory segment ξ starting in $\mathbf{x}$, that reaches some point in I, such that $\mathsf{Sig}(\xi)$ is a suffix of $e_2 \ldots e_k e_1$. It is easy to see that $\overline{\mathsf{Pre}}_\sigma(I)$ is a polygonal subset of the plane which can be calculated using the following procedure. First define

$$\overline{\mathsf{Pre}}_e(I) = \{\mathbf{x} \mid \exists \xi : [0, t] \rightarrow \mathbb{R}^2, t > 0 \,.\, \xi(0) = \mathbf{x} \wedge \xi(t) \in I \wedge \mathsf{Sig}(\xi) = e\}$$

and apply this operation k times: $\overline{\mathsf{Pre}}_\sigma(I) = \bigcup_{i=1}^{k} \overline{\mathsf{Pre}}_{e_i}(I_i)$ with $I_1 = I$, $I_k = \mathsf{Pre}_{e_k, e_1}(I_1)$ and $I_i = \mathsf{Pre}_{e_i, e_{i+1}}(I_{i+1})$, for $2 \leq i \leq k-1$.

[3] We do not define the viability kernel to be closed as in [2].

[4] Notice that this theorem can be used to compute $\mathsf{Viab}(I)$ for any $I \subseteq e_1$.

Now, let

$$K_\sigma = \bigcup_{i=1}^{k} (int(P_i) \cup e_i) \tag{2}$$

where P_i is such that $e_{i-1} \in in(P_i)$, $e_i \in out(P_i)$ and $int(P_i)$ is the interior of P_i.

Theorem 3. *If σ is not DIE,* $\mathsf{Viab}(K_\sigma) = \overline{\mathsf{Pre}}_\sigma(S)$*, otherwise* $\mathsf{Viab}(K_\sigma) = \emptyset$*.*

This result provides a non-iterative algorithmic procedure for computing the viability kernel of K_σ.

Example 7. Let $\sigma = e_1 \dots e_8 e_1$. Fig. 3 depicts: (a) K_σ, and (b) $\overline{\mathsf{Pre}}_\sigma(S)$ ∎

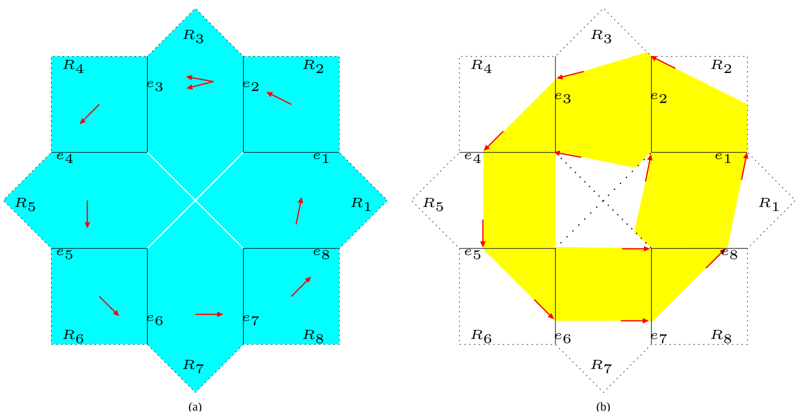

Fig. 3. Viability kernel.

3.2 Controllability kernel

We say $K \subset \mathbb{R}^2$ is *controllable* if for any two points $\mathbf{x}$ and $\mathbf{y}$ in K there exists a trajectory segment ξ starting in $\mathbf{x}$ that reaches an arbitrarily small neighborhood of $\mathbf{y}$ without leaving K. More formally, K is controllable iff $\forall \mathbf{x}, \mathbf{y} \in K, \forall \delta > 0, \exists \xi : [0,t] \to \mathbb{R}^2, t > 0 \;.\; (\xi(0) = \mathbf{x} \wedge |\xi(t) - \mathbf{y}| < \delta \wedge \forall t' \in [0,t] \;.\; \xi(t') \in K)$. The *controllability kernel* of K, denoted $\mathsf{Cntr}(K)$, is the largest controllable subset of K. The same notions can be defined for the discrete dynamical system $\mathcal{D}_\sigma$.

Define

$$\mathcal{C}_\mathcal{D}(\sigma) = \begin{cases} \langle L, U \rangle & \text{if } \sigma \text{ is EXIT-BOTH} \\ \langle L, u^* \rangle & \text{if } \sigma \text{ is EXIT-LEFT} \\ \langle l^*, U \rangle & \text{if } \sigma \text{ is EXIT-RIGHT} \\ \langle l^*, u^* \rangle & \text{if } \sigma \text{ is STAY} \\ \emptyset & \text{if } \sigma \text{ is DIE} \end{cases} \tag{3}$$

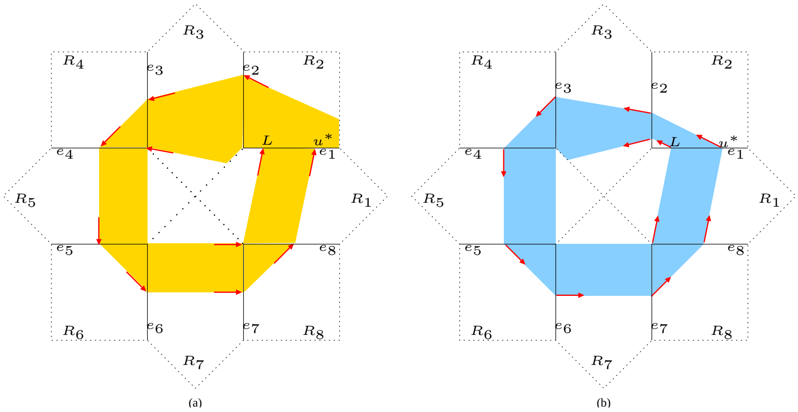

Fig. 4. Predecessors and successors of a simple cycle.

Theorem 4. *For $\mathcal{D}_\sigma$, $\mathcal{C}_\mathcal{D}(\sigma) = \mathsf{Cntr}(S)$.*

For $I \subseteq e_1$ let us define $\overline{\mathsf{Succ}}_\sigma(I)$ as the set of all points $\mathbf{y} \in \mathbb{R}^2$ for which there exists a trajectory segment ξ starting in some point $x \in I$, that reaches $\mathbf{y}$, such that $\mathsf{Sig}(\xi)$ is a prefix of $e_1 \dots e_k$. The successor $\overline{\mathsf{Succ}}_\sigma(I)$ is a polygonal subset of the plane which can be computed similarly to $\overline{\mathsf{Pre}}_\sigma(I)$.

Example 8. Let $\sigma = e_1 \cdots e_8 e_1$. Fig. 4 depicts: (a) $\overline{\mathsf{Pre}}_\sigma(L, u^*)$, (b) $\overline{\mathsf{Succ}}_\sigma(L, u^*)$, with $L = \frac{1}{5} < u^* = \frac{23}{30}$. ∎

Define

$$\mathcal{C}(\sigma) = (\overline{\mathsf{Succ}}_\sigma \cap \overline{\mathsf{Pre}}_\sigma)(\mathcal{C}_\mathcal{D}(\sigma)) \tag{4}$$

Theorem 5. $\mathcal{C}(\sigma) = \mathsf{Cntr}(K_\sigma)$.

This result provides a non-iterative algorithmic procedure for computing the controllability kernel of K_σ.

Example 9. Let $\sigma = e_1 \cdots e_8 e_1$. Recall that σ is EXIT-LEFT with $L = \frac{1}{5} < u^* = \frac{23}{30}$. Fig. 5(a) depicts $\mathsf{Cntr}(K_\sigma)$. ∎

Convergence. A trajectory ξ *converges* to a set $K \subset \mathbb{R}^2$ if $\lim_{t\to\infty} dist(\xi(t), K) = 0$. For $\mathcal{D}_\sigma$, convergence is defined as $\lim_{n\to\infty} dist(\xi_n, I) = 0$. The following result says that the controllability kernel $\mathcal{C}_\mathcal{D}(\sigma)$ can be considered to be a kind of (weak) limit cycle of $\mathcal{D}_\sigma$.

Theorem 6. *For $\mathcal{D}_\sigma$, any viable trajectory in S converges to $\mathcal{C}_\mathcal{D}(\sigma)$.*

Furthermore, $\mathcal{C}(\sigma)$ can be regarded as a (weak) limit cycle of the SPDI. The following result is a direct consequence of Theorem 3 and Theorem 6.

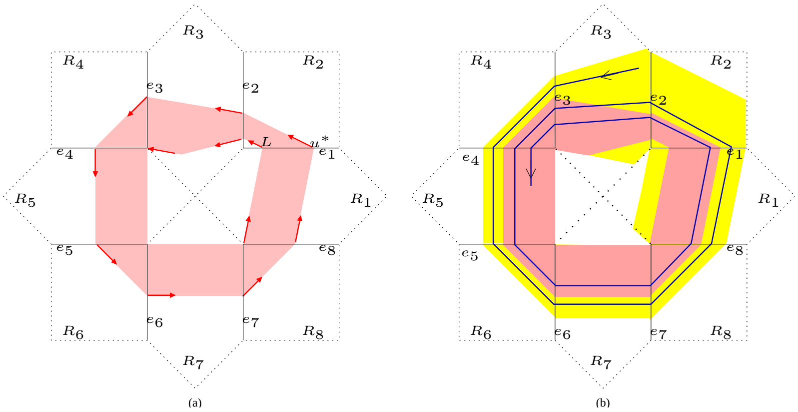

Fig. 5. Controllability kernel of a simple cycle.

Theorem 7. *Any viable trajectory in K_σ converges to $\mathcal{C}(\sigma)$.*

Example 10. Fig. 5(b) shows a trajectory with signature $\sigma = e_1 \cdots e_8 e_1$ which is viable in K_σ and converges to $\mathcal{C}(\sigma)$. ■

STAY cycles. The controllability kernels of STAY-cycles have stronger limit cycle properties. We say that K is *invariant* if for any $x \in K$, every trajectory starting in x is viable in K. The following result is a corollary of the previous theorems.

Corollary 1. *Let σ be STAY. Then,*
(1) $\mathcal{C}(\sigma)$ is invariant.
(2) There exists a neighborhood K of $\mathcal{C}(\sigma)$ such that any viable trajectory starting in K converges to $\mathcal{C}(\sigma)$.

Fixpoints. Here we give an alternative characterization of the controllability kernel of a cycle in SPDI. As in [7], let us call a point x in e_1 a *fixpoint* iff $x \in \mathsf{Succ}_\sigma(x)$. We call a point $\mathbf{x} \in K_\sigma$ a *periodic point* iff there exists a trajectory segment ξ starting and ending in $\mathbf{x}$, such that $\mathsf{Sig}(\xi)$ is a cyclic shift of σ (hence, there exists also an infinite periodic trajectory passing through x). The following result is a corollary of the previous theorems and definitions.

Corollary 2. *For SPDIs,*
(1) $\mathcal{C}_\mathcal{D}(\sigma)$ is the set of all the fixpoints in e_1.
(2) $\mathcal{C}(\sigma)$ is the set of all the periodic points in K_σ.

4 Phase portrait

Let ξ be any trajectory without self-crossings. Recall that ξ is assumed to have an infinite signature. An immediate consequence of the results proven in [1] is that $\mathsf{Sig}(\xi)$ can be canonically expressed as a sequence of edges and cycles of the form $r_1 s_1^* \ldots r_n s_n^\omega$, where

1. For all $1 \leq i \leq n$, r_i is a sequence of pairwise different edges, and s_i is a simple cycle.
2. For all $1 \leq i \neq j \leq n$, r_i and r_j are disjoint, and s_i and s_j are different.
3. For all $1 \leq i \leq n-1$, s_i is repeated a finite number of times.
4. s_n is repeated forever.

Hence,

Theorem 8. *Every trajectory with infinite signature which does not have self-crossings converges to the controllability kernel of some simple edge-cycle.*

Corollary 3. *1. Any trajectory ξ with infinite signature without self-crossings is such that its limit set* limit(ξ) *is a subset of the controllability kernel $\mathcal{C}(\sigma)$ of a simple edge-cycle σ.*
2. Any point in $\mathcal{C}(\sigma)$ is a limit point of a trajectory ξ with infinite signature without self-crossings

We conclude that controllability kernels are important elements of the phase portrait of an SPDI yielding an analog of Poincaré-Bendixson theorem for simple trajectories. Moreover, all such components of the phase portrait can be algorithmically constructed. Indeed, since there are finitely many simple cycles, the following algorithm computes all the limit sets and their attraction basins for such kind of trajectories:

$$\textbf{for each } \text{simple cycle } \sigma \textbf{ compute } \mathcal{C}(\sigma), \overline{\mathsf{Pre}}_\sigma(S)$$

Example 11. Fig. 6 shows an SPDI with two edge cycles $\sigma_1 = e_1, \cdots, e_8, e_1$ and $\sigma_2 = e_{10}, \cdots, e_{15}, e_{10}$, and their respective controllability kernels. Every simple trajectory eventually arrives (or converges) to one of the two limit sets and rotates therein forever. ■

Self-crossing trajectories. Actually, the previous example illustrates the difficulties that arise when exploring the limit behavior of self-crossing trajectories of an SPDI. The figure shows that there exist infinite self-crossing (and even periodic) trajectories that keep switching between the two cycles forever. In this particular case, it can be shown that all trajectories converge to the "joint controllability kernel" $\mathsf{Cntr}(K_{\sigma_1} \cup K_{\sigma_2})$ which turns out to be $\mathcal{C}(\sigma_1) \cup \mathcal{C}(\sigma_2)$[5]. However, the analysis of limit behaviors of self-cutting trajectories in the general case is considerably more difficult and challenging.

[5] The cross-shaped region is the bridge between the two cycles.

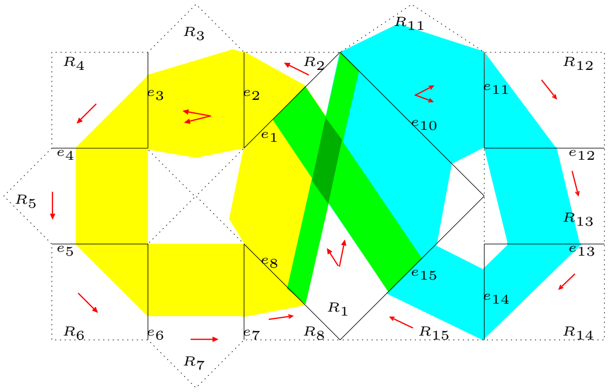

Fig. 6. Another SPDI and its "phase-portrait".

5 Concluding remarks

The contribution of this paper is an automatic procedure to analyze the qualitative behavior of non-deterministic planar hybrid systems. Our algorithm enumerates all the "limit cycles" (i.e., controllability kernels) and their local basins of attraction (i.e., viability kernels).

Our analysis technique for a single cycle is very similar to the one used in [7] for n-dimensional systems. However, for polygonal systems, we are able to prove further properties such as controllability of and convergence to the set of fixpoints, and that there are only a finite number of them. These results are the analog of Poincaré-Bendixson for polygonal differential inclusions. The difference with [9] is that our results hold for non-deterministic systems.

This work is a first step in the direction of finding an algorithm for automatically constructing the complete phase portrait of an SPDI. This would require identifying and analyzing other useful structures such as stable and unstable manifolds, orbits (generated by identity Poincaré maps), bifurcation points (resulting of the non-deterministic qualitative behavior at the vertices of the polygons), limit behaviors of self-intersecting trajectories, etc.

We are currently developing a tool for qualitative analysis of SPDIs. The tool already implements the reachability algorithm published in [1] as well as most of the basic functionalities required for constructing the phase portrait. We have used it to analyze (though not completely automatically) all the examples of this paper.

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A Appendix

Let $\sigma = e_1 \cdots e_k e_1$ be a simple edge-cycle, i.e., $e_i \neq e_j$ for all $1 \leq i \neq j \leq k$. Let $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$ with $F = \langle f_l, f_u \rangle$ (we suppose that this representation is normalized). We denote by $\mathcal{D}_\sigma$ the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$. For $x \in \mathbb{R}$ and $I \subset \mathbb{R}$, $x < I$ means that $x < y$ for all $y \in I$.

Theorem 2 *For $\mathcal{D}_\sigma$, if σ is not DIE then* $\mathsf{Viab}(e_1) = S$, *otherwise* $\mathsf{Viab}(e_1) = \emptyset$.
Proof. If σ is DIE, $\mathcal{D}_\sigma$ has no trajectories. Therefore, $\mathsf{Viab}(e_1) = \emptyset$.
Let σ be not DIE. We first prove that any viability domain is a subset of S: Let I be a viability domain. Then, for all $x \in I$, there exists a trajectory starting in x which is viable in I. Then, $x \in \mathsf{DomSucc}_\sigma = S$. Thus, $I \subseteq S$.
Now, let us prove that S is a viability domain: It suffices to show that for all $x \in S$, $\mathsf{Succ}_\sigma(x) \cap S \neq \emptyset$.
Let $x \in S$.
If σ is STAY, we have that both l^* and u^* belong to $S \cap J$. It follows that both $f_l(x)$ and $f_u(x)$ are in S.
If σ is EXIT-LEFT, we have that $l^* < S \cap J$ and $u^* \in S \cap J$. Then, $f_u(x) \in S$.
If σ is EXIT-RIGHT, we have that $l^* \in S \cap J$ and $u^* > S \cap J$. Then, $f_l(x) \in S$.
If σ is EXIT-BOTH, we have that $l^* < S \cap J$ and $u^* > S \cap J$. If $x \in J$: then $x \in F(x)$. If $x < J$: then $f_l(x) < x < S \cap J$, and either $f_u(x) \in S \cap J$ or $f_u(x) > S \cap J$ (the other case yields a contradiction). If $x > J$: similar to the previous case.
Thus, for all $x \in S$, $\mathsf{Succ}_\sigma(x) \cap S \neq \emptyset$.
Hence, $\mathsf{Viab}(e_1) = S$. □

Theorem 3 *If σ is not DIE,* $\mathsf{Viab}(K_\sigma) = \overline{\mathsf{Pre}}_\sigma(S)$*, otherwise* $\mathsf{Viab}(K_\sigma) = \emptyset$.
Proof: If σ is DIE, trivially $\mathsf{Viab}(K_\sigma) = \emptyset$.
Let σ be not DIE. We first prove that any viability domain K, with $K \subseteq K_\sigma$, is a subset of $\overline{\mathsf{Pre}}_\sigma(S)$: Let $\mathbf{x} \in K$. Then, there exists a trajectory ξ such that $\xi(0) = \mathbf{x}$ and for all $t \geq 0$, $\xi(t) \in K$. Clearly, the sequence $x_1 x_2 \ldots$ of the intersections of ξ with e_1 is a trajectory of $\mathcal{D}_\sigma$. Then, by Theorem 2, $x_i \in S$ for all $i \geq 1$. Thus, $\mathbf{x} \in \overline{\mathsf{Pre}}_\sigma(S)$.
It remains to prove that $\overline{\mathsf{Pre}}_\sigma(S)$ is a viability domain. Let $\mathbf{x} \in \overline{\mathsf{Pre}}_\sigma(S)$. Then, there exists a trajectory segment $\bar{\xi} : [0, T] \to \mathbb{R}^2$ such that $\bar{\xi}(T) \in S$ and $\mathsf{Sig}(\bar{\xi})$ is a suffix of σ. Theorem 2 implies that $\bar{\xi}(T)$ is the initial state of some trajectory ξ with $\mathsf{Sig}(\xi) = \sigma^\omega$. It is straightforward to show that for all $t \geq 0$, $\xi(t) \in \overline{\mathsf{Pre}}_\sigma(S)$. Concatenating $\bar{\xi}$ and ξ, we obtain a viable trajectory starting in $\mathbf{x}$.
Hence, $\mathsf{Viab}(K_\sigma) = \overline{\mathsf{Pre}}_\sigma(S)$. □

Theorem 4 *For $\mathcal{D}_\sigma$, $\mathcal{C}_\mathcal{D}(\sigma) = \mathsf{Cntr}(S)$.*
Proof. Controllability of $\mathcal{C}_\mathcal{D}(\sigma)$ follows from the reachability result in [1]. To prove that $\mathcal{C}_\mathcal{D}(\sigma)$ is maximal we reason by contradiction. Suppose it is not. Then, there should exist a controllable set $C \supset \mathcal{C}_\mathcal{D}(\sigma)$. Since $C \subseteq S \cap J$, there should exist $y \in C$ such that either $y < l^*$, or $y > u^*$. In any case, controllability implies that for all $l^* < x < u^*$, there exists a trajectory segment starting in x that reaches an arbitrarily small neighborhood of y. From [1] we know that $\mathit{Reach}(x) \subset (l^*, u^*)$, which yields a contradiction. Hence, $\mathcal{C}_\mathcal{D}(\sigma)$ is the controllability kernel of S. □

Theorem 5 $\mathcal{C}(\sigma) = \mathsf{Cntr}(K_\sigma)$.
Proof. Let $\mathbf{x}, \mathbf{y} \in \mathcal{C}(\sigma)$. Since $\mathbf{y} \in \overline{\mathsf{Succ}}_\sigma(\mathcal{C}_\mathcal{D}(\sigma))$, there exists a trajectory segment starting in some point $w \in \mathcal{C}_\mathcal{D}(\sigma)$ and ending in $\mathbf{y}$. Let ϵ be an arbitrarily small number and $B_\epsilon(\mathbf{y})$ be the set of all points $\mathbf{y}'$ such that $|\mathbf{y} - \mathbf{y}'| < \epsilon$. Clearly, $w \in \overline{\mathsf{Pre}}_\sigma(B_\epsilon(\mathbf{y})) \cap \mathcal{C}_\mathcal{D}(\sigma)$. Now, since $\mathbf{x} \in \overline{\mathsf{Pre}}_\sigma(\mathcal{C}_\mathcal{D}(\sigma))$, there exists a trajectory segment starting in $\mathbf{x}$ and ending in some point $z \in \mathcal{C}_\mathcal{D}(\sigma)$. Since $\mathcal{C}_\mathcal{D}(\sigma)$ is controllable, there exists a trajectory segment starting in z that reaches a point in $\overline{\mathsf{Pre}}_\sigma(B_\epsilon(\mathbf{y})) \cap \mathcal{C}_\mathcal{D}(\sigma)$. Thus, there is a trajectory segment that starts in $\mathbf{x}$ and ends in $B_\epsilon(\mathbf{y})$. Therefore, $\mathcal{C}(\sigma)$ is controllable. Maximality follows from the maximality of $\mathcal{C}_\mathcal{D}(\sigma)$ and the definition of $\overline{\mathsf{Succ}}_\sigma$ and $\overline{\mathsf{Pre}}_\sigma$. Hence, $\mathcal{C}(\sigma)$ is the controllability kernel of K_σ. □

Theorem 6 *For $\mathcal{D}_\sigma$, any viable trajectory in S converges to $\mathcal{C}_\mathcal{D}(\sigma)$.*
Proof. Let $x_1 x_2 \ldots$ a viable trajectory. Clearly, $x_i \in S \cap J$ for all $i \geq 2$. Recall that $\mathcal{C}_\mathcal{D}(\sigma) \subseteq S \cap J$. There are three cases: (1) There exists $N \geq 2$ such that $x_N \in \mathcal{C}_\mathcal{D}(\sigma)$. Then, for all $n \geq N$, $x_n \in \mathcal{C}_\mathcal{D}(\sigma)$. (2) For all n, $x_n < \mathcal{C}_\mathcal{D}(\sigma)$. Therefore, $x_n < l^*$. Let $\hat{x}_n$ be such that $\hat{x}_1 = x_1$ and for all $n \geq 1$, $\hat{x}_{n+1} = f_l(\hat{x}_n)$. Clearly, for all n, $\hat{x}_n \leq x_n < l^*$, and $\lim_{n\to\infty} \hat{x}_n = l^*$, which implies $\lim_{n\to\infty} x_n = l^*$. (3) For all n, $x_n > \mathcal{C}_\mathcal{D}(\sigma)$. Therefore, $u^* < x_n$. Let $\hat{x}_n$ be such that $\hat{x}_1 = x_1$ and for all $n \geq 1$, $\hat{x}_{n+1} = f_u(\hat{x}_n)$. Clearly, for all n, $u^* < x_n \leq \hat{x}_n$, and $\lim_{n\to\infty} \hat{x}_n = u^*$, which implies $\lim_{n\to\infty} x_n = u^*$. Hence, $x_1 x_2 \ldots$ converges to $\mathcal{C}(\sigma)$. □