On Weyl's theorem

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A well-known theorem of H. Weyl [2] asserts the uniform distribution mod. 1 of the sequence $p(n)$ where $p(t) = a_0 t^k + \ldots + a_k$ is any polynomial with a_0 irrational. In [1] is presented a proof of this theorem based only the mean ergodic theorem. Reading constructively this proof, we get an even shorter direct proof.

To simplify, we present only the argument for $p(n) = \alpha n^2$, α irrational. But the same argument holds in general.

We let E be the vector space of all bounded sequence $x = (x(n))$ of complex numbers. We define $\mu_n(x) = 1/n\Sigma_{i<n}x(i)$ and we let E_0 be the subset of sequences x such that $\mu_n(x)$ converges to 0. On E we have also a product operation $(xy)(n) = x(n)y(n)$ and a conjugate operation $\overline{x}(n) = \overline{x(n)}$ that are defined pointwise. We have also the shift operation $S : E \to E$ defined by $S(x)(n) = x(n+1)$. Finally we identify a complex number r with the constant sequence $r(n) = r$.

Since $\mu_k((x-a)\overline{(x-a)}) \geq 0$ for all a and hence for $a = \mu_k(x)$ we get the Cauchy-Schwartz inequality $|\mu_k(x)|^2 \leq \mu_k(x\overline{x})$ for all $x \in E$.

Let α be a fixed irrational number and let $x(n) = e^{2\pi i \alpha n^2}$. Fix $\epsilon > 0$. We have to show that $|\mu_k(x)| \leq 2\epsilon$ for k large enough.

We remark first that we have $x - S^l(x) \in E_0$ and $S^i(x)\overline{S^j(x)} \in E_0$ for $i \neq j$ since this sequence is a multiple of a sequence of the form $u(n) = e^{2\pi i \alpha l n}$ with $l \neq 0$.

We fix N such that $1/N \leq \epsilon^2/2$.

We let x_N be $1/N\Sigma_{i<N}S^i(x)$, and we notice that we have

$$x_N\overline{x_N} = 1/N^2\Sigma_{i,j}S^i(x)\overline{S^j(x)} = 1/N^2\Sigma_{i\neq j}S^i(x)\overline{S^j(x)} + \ 1/N$$

It follows that $x_N\overline{x_N} - 1/N \in E_0$, and so $\mu_k(x_N\overline{x_N}) \leq \epsilon^2$ for k large enough. Hence, by Cauchy-Schwartz inequality, $|\mu_k(x_N)| \leq \epsilon$ for k large enough. Since we have $x - x_N \in E_0$ it follows also that we have $|\mu_k(x)| \leq 2\epsilon$ for k large enough, as desired.

References

[1] J. Cigler. On a theorem of H. Weyl Composition Mathematoca, Vol.21, p. 151-154 (1969).

[2] H. Weyl Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann. 77 (1916).