A Boolean Model of Ultrafilters

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Abstract

We introduce the notion of Boolean measure algebra. It can be described shortly using some standard notations and terminology. If B is any Boolean algebra, let B^N denote the algebra of sequences $(x_n),\ x_n \in B$. Let us write $p_k \in B^N$ the sequence such that $p_k(i) = 1$ if $i \leq k$ and $p_k(i) = 0$ if $k < i$. If $x \in B$, denote by $x^* \in B^N$ the constant sequence $x^* = (x, x, x, \ldots)$. We define a *Boolean measure algebra* to be a Boolean algebra B with an operation $\mu : B^N \rightarrow B$ such that $\mu(p_k) = 0$ and $\mu(x^*) = x$.

Any Boolean measure algebra can be used to model non principal ultrafilters in a suitable sense. Also, we can build effectively the initial Boolean measure algebra. This construction is related to the closed open Ramsey Theorem [8].

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Introduction

Non-principal ultrafilters over natural numbers constitute a typical example of objects that cannot be described effectively. Their first appearence, as two-valued additive measures on the set of natural numbers such that each singleton has measure zero, may be traced back to Ulam and Tarski [19, 17]. Soon after, Sierpinski showed that the existence of such a measure yields the existence of a set which is not Lebesgue-measurable [16], a result that illustrates well the non-effective character of such objects. It is in some sense surprising that, despite this non-effectiveness, these objects can be used in the proof of concrete statements; for instance a quite perspicuous proof of Ramsey's theorem [9] uses such an ultrafilter as an "oracle" deciding what subsets of natural numbers are "large". We show that, if we replace standard truth values by opens of a suitable formal space (in this case regular or σ-complete ideals of a Boolean algebra), it is possible to describe effectively a non-principal ultrafilter.

It is possible to describe shortly the gist of the paper, using the following notations and terminology. Let F_2 be the Boolean algebra with two elements. If B is any Boolean algebra, let B^N denote the algebra of sequences $(x_n),\ x_n \in B$. Let us write $p_k \in B^N$ the sequence such that $p_k(i) = 1$ if $i \leq k$ and $p_k(i) = 0$ if $k < i$. If $x \in B$, denote by $x^* \in B^N$ the constant sequence $x^* = (x, x, x, \ldots)$. We define a *Boolean measure algebra* to be a Boolean algebra B with an operation $\mu : B^N \rightarrow B$ such that $\mu(p_k) = 0$ and $\mu(x^*) = x$. It is clear that a Boolean measure algebra structure on F_2 is another description of a non principal ultrafilter, and so, as explained above, cannot be built effectively. We show that *any* Boolean measure algebra can be used to model non-principal ultrafilters and we build *effectively* the initial Boolean measure algebra. This construction reveals unexpected connections with the closed open Ramsey theorem [8].

Our meta-language is kept informal, but is quite similar to the one used in Bishop's book [4]. We think that our development can be rather directly formalised in a setting like constructive type theory [15].

1 Initial Boolean Measure Algebra

1.1 Basic Notions

We shall follow closely the terminology and notations of [10]. A Boolean algebra will be a ring B in which every element is idempotent. We write $x + y$ for the ring addition (exclusive-or) and $x.y$ or xy for the ring multiplication. We can then define $x \vee y = x + y + xy$.

1.2 The Boolean algebra C

We let C be the initial Boolean algebra with an infinitary operation $\mathsf{C}^N \rightarrow \mathsf{C}$. It is standard that this operation has an inverse $\mathsf{C} \rightarrow \mathsf{C}^N$. We write $x \longmapsto x(n)$ the n-th component of this inverse. Since $\mathsf{C}^N \rightarrow \mathsf{C}$ is one-to-one, we write simply $(x_0, x_1, x_2, \ldots) \in \mathsf{C}$ the image of an infinite sequence $(x_0, x_1, x_2, \ldots) \in \mathsf{C}^N$. Using this convention, we consider that $x^* \in \mathsf{C}$ if $x \in \mathsf{C}$. In general, we have thus $(x_0, x_1, x_2, \ldots)(n) = x_n$ and $x = (x(0), x(1), x(2), \ldots)$. It follows also from initiality that the following induction principle is valid for C :

Proposition 1: If $X \subseteq \mathsf{C}$ is such that $0 \in X,\ 1 \in X$ and $(x_n) \in X$ whenever $x_n \in X$ for all n, then $X = \mathsf{C}$.

1.3 Another description of C

The Boolean algebra C can be seen as a formal description of the set $[\mathcal{N} \rightarrow F_2]$ of all Boolean continuous functions over the Baire space $\mathcal{N}$, whose elements are infinite sequences $\alpha = n_0 n_1 n_2 \ldots$ of natural numbers. (This is equivalent to the Boolean algebra of closed open subsets of the Baire space.) Any element $x \in \mathsf{C}$ defines a Boolean function $\phi(x) \in [\mathcal{N} \rightarrow F_2]$ by taking $\phi(0)(\alpha) = 0$, $\phi(1)(\alpha) = 1$ and $\phi((x_n))(n_0 n_1 n_2 \ldots) = \phi(x_{n_0})(n_1 n_2 \ldots)$. [1]

1.4 An ideal of C

The subset $\mathsf{C}_0 \subseteq \mathsf{C}$ is defined inductively by the following rules [1]

$$\frac{}{0 \in \mathsf{C}_0} \qquad \frac{\ldots \quad xx(n) \in \mathsf{C}_0 \quad \ldots \ (n \geq N)}{x \in \mathsf{C}_0}.$$

Lemma 1: If $x \in \mathsf{C}_0$ and $y \leq x$ then $y \in \mathsf{C}_0$.

Proof: By induction on the proof that $x \in \mathsf{C}_0$. If $x = 0$ then $y = 0$ and hence $y \in \mathsf{C}_0$. Otherwise there exists N such that $xx(n) \in \mathsf{C}_0$ for all $n \geq N$. By induction hypothesis, $yy(n) \in \mathsf{C}_0$ for all $n \geq N$ since $yy(n) \leq xx(n)$. Hence $y \in \mathsf{C}_0$. $\square$

Lemma 2: If $ux \in \mathsf{C}_0$ and $v(1-x) \in \mathsf{C}_0$ then $uv \in \mathsf{C}_0$.[2]

Proof: Let us say that $x \in \mathsf{C}$ is *eliminable* iff $ux \in \mathsf{C}_0$ and $v(1-x) \in \mathsf{C}_0$ imply $uv \in \mathsf{C}_0$. Lemma 2 states that all elements are eliminable. We prove this using proposition 1. It is clear using lemma 1 that 1 and 0 are eliminable. Suppose that x_n is eliminable for all n. We let x be $x = (x_n)$ and we show that if $ux \in \mathsf{C}_0$ and $v(1-x) \in \mathsf{C}_0$ then $uv \in \mathsf{C}_0$ by induction on the proof that $ux \in \mathsf{C}_0$ and $v(1-x) \in \mathsf{C}_0$.

If $ux = 0$ then $u \leq 1 - x$ and hence $uv \leq v(1-x)$. The conclusion follows from lemma 1.

If $v(1-x) = 0$ then $v \leq x$ and hence $uv \leq ux$. The conclusion follows from lemma 1.

If there exists N such that $uxu(n)x(n) \in \mathsf{C}_0$ and $v(1-x)v(n)(1-x(n)) \in \mathsf{C}_0$ for all $n \geq N$ then by induction hypothesis we have that

- $uxu(n)x(n) \in \mathsf{C}_0$ and $v(1-x) \in \mathsf{C}_0$ implies $uvu(n)x(n) \in \mathsf{C}_0$ and
- $ux \in \mathsf{C}_0$ and $v(1-x)v(n)(1-x(n)) \in \mathsf{C}_0$ implies $uvv(n)(1-x(n)) \in \mathsf{C}_0$ for all $n \geq N$.

Since all $x(n)$ are eliminable this implies $uvu(n)v(n) \in \mathsf{C}_0$ for all $n \geq N$ and hence $uv \in \mathsf{C}_0$ as desired.

Hence, if x_n are eliminable for all n then so is (x_n). By proposition 1, this shows that all elements of C are eliminable. $\square$

Theorem 1: C_0 is an ideal of C.

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[1]The fact that any continuous map $[\mathcal{N} \rightarrow F_2]$ are of the form $\phi(x)$ for one $x \in \mathsf{C}$ can be proved using a classical metalanguage, or using some form of bar induction [18].

[2]This is similar to cut-elimination results for infinitary logic; see for instance [13].

Proof: Using lemma 1, it is enough to show that C_0 is closed by binary sup. If $x \in \mathsf{C}_0$ and $y \in \mathsf{C}_0$ then $1x = x \in \mathsf{C}_0$ and $(x \vee y)(1-x) = y \in \mathsf{C}_0$. By lemma 2, we have that $x \vee y = 1(x \vee y) \in \mathsf{C}_0$. $\square$

Corollary: $x - x^* \in \mathsf{C}_0$ for all x.

Proof: We have directly $y = x(1-x^*) \in \mathsf{C}_0$ and $z = x^*(1-x) \in \mathsf{C}_0$, because $y(n)y = z(n)z = 0$ for all n. Hence $x - x^* = y \vee z \in \mathsf{C}_0$ by theorem 1. $\square$

It seems difficult to prove directly that $x - x^* \in \mathsf{C}_0$ for all x, without first proving that C_0 is closed under sum.

Proposition 2: If $x_n \in \mathsf{C}_0$ for $n \geq N$ then $(x_0, x_1, \dots) \in \mathsf{C}_0$.

Proof: Let x be $(x_0, x_1, \dots)$. By lemma 1, we have $xx_n \in \mathsf{C}_0$ for $n \geq N$ and hence $x \in \mathsf{C}_0$. $\square$

Let B the quotient Boolean algebra C/C_0. By proposition 2, we can quotient the operation $\mathsf{C}^N \to \mathsf{C}$ to get an operation $\mu : \mathsf{B}^N \to \mathsf{B}$. Furthermore this operation satisfies $\mu(p_k) = 0$ and $\mu(x^*) = x$ for all $x \in \mathsf{B}$. Thus $\mathsf{B}, \mu : \mathsf{B}^N \to \mathsf{B}$ is a Boolean measure algebra.

Theorem 2: $\mathsf{B}, \mu : \mathsf{B}^N \to \mathsf{B}$ is the initial Boolean measure algebra.

Proof: Let $A, \mu : A^N \to A$ be any Boolean measure algebra. By initiality of C there exists a unique morphism $g : \mathsf{C} \to A$ such that $g(\alpha) = \mu(g \circ \alpha)$ for all $\alpha \in \mathsf{C}^N$. It is then direct to show that $x \in \mathsf{C}_0$ implies $g(x) = 0$ by induction on the proof that $x \in \mathsf{C}_0$: if $g(xx(n)) = 0$ for all $n \geq N$, then identifying x and $(x(0), x(1), \dots) \in \mathsf{C}^N$ we have $g(x) = \mu(g \circ x)$ and hence

$$g(x) = \mu(g(x)^*)g(x) = \mu(g \circ x^*)\mu(g \circ x) = \mu(g \circ x^* x) = 0.$$

It follows that we can factorize the morphism g to a morphism $f : \mathsf{B} \to A$ which satisfies

$$f(\mu(\alpha)) = \mu(f \circ \alpha)$$

for all $\alpha \in \mathsf{B}^N$. This reasoning shows also the uniqueness of such a morphism. $\square$

1.5 A finite version

In a similar way, we can analyse *binary* measure algebras, that are Boolean algebras B with a morphism $\mu : B^2 \to B$ such that $\mu(x, x) = x$ for all $x \in B$. The initial binary Boolean measure algebra can be directly described as the Boolean algebra $\mathsf{B} = F_2^2$ with the operation $\mu((x_0, x_1), (y_0, y_1)) = \mu(x_0, y_1)$. This follows from the following result.

Proposition: In B we have $\mu(\mu(x_0, x_1), \mu(y_0, y_1)) = \mu(x_0, y_1)$.

Proof: We first show $\mu(\mu(1,0), \mu(0,1)) = 1$. This follows from $\mu(\mu(1,0), 1) = \mu(1, \mu(0,1)) = 1$: we have $\mu(0,1) = \mu(\mu(0,1), \mu(0,1))$ and hence

$$\mu(0,1) = \mu(0, \mu(0,1)).$$

We have also

$$\mu(1,0) = \mu(\mu(1,0), \mu(1,0)).$$

Hence $1 = \mu(1,1) = \mu(\mu(1,0), 1)$. The equality $\mu(1, \mu(0,1)) = 1$ is proved similarly.

It follows that

$$\mu(\mu(x_0, x_1), \mu(y_0, y_1)) = \mu(\mu(x_0, 0), \mu(0, y_1))$$

and also

$$\mu(x_0, y_1) = \mu(\mu(x_0, x_0), \mu(y_1, y_1)) = \mu(\mu(x_0, 0), \mu(0, y_1)).$$

Hence the result. $\square$

Alternatively, we can build B as a quotient of the initial Boolean algebra with an operation $\mathsf{C}^2 \to \mathsf{C}$ (which is isomorphic to the Lindenbaum algebra of propositional logic). We define $\mathsf{C}_0 \subseteq \mathsf{C}$ by the rules

$$\frac{}{0 \in \mathsf{C}_0} \qquad \frac{xx(0) \in \mathsf{C}_0 \quad xx(1) \in \mathsf{C}_0}{x \in \mathsf{C}_0}.$$

We can then prove as in the infinitary case:

Theorem: C_0 is a ideal of C.

In this case C/C_0 is isomorphic to F_2^2.
Similarly, we can define n-ary Boolean measure algebra, and prove that the initial n-ary Boolean measure algebra is F_2^n.

2 How to Interpret Ultrafilters

2.1 Properties of Boolean Measure Algebra

Let $B, \mu : B^N \to B$ be a Boolean measure algebra.

Proposition 3: B is a *σ-algebra* [10]; any sequence of element of B has a supremum.

Proof: Let y_n be $x_0 \vee \ldots \vee x_n$. We have $x_k \leq y_n$ for $k \leq n$ and hence $\mu(x_k^*) = x_k \leq \mu(y_0, y_1, \ldots)$. If $x_n \leq z$ for all n then $y_n \leq z$ for all n and hence $\leq \mu(y_0, y_1, \ldots) \leq z = \mu(z^*)$. □

Proposition 4: If $f \in B^N$ we have $\mu(f) \leq \bigvee_{N \leq n} f(n)$.

Proof: Let $g \in B^N$ be defined by $g(n) = 0$ if $n < N$ and $g(n) = f(N) \vee \ldots \vee f(n)$ for $n \geq N$. The proof of proposition 3 shows that $\bigvee_{N \leq n} f(n) = \mu(g)$. Also $f(n) \leq g(n)$ for $n \geq N$ and hence $\mu(f) \leq \mu(g)$. □

2.2 A Proof of Ramsey's Theorem

We use the previous results to interpret a proof which uses a non principal ultrafilter, using *any* Boolean measure algebra B. This is an instance of a proof of Ramsey's theorem presented in [9]. Let $\chi \in [N \to N \to 2]$ be given. We write χ_n the function $m \longmapsto \chi(n)(m)$. We show

$$\phi \equiv \exists n_1, n_2, n_3 \; [n_1 < n_2 < n_3 \;\wedge \chi_{n_1}(n_2) = \chi_{n_1}(n_3) = \chi_{n_2}(n_3)]$$

using a non principal ultrafilter μ.[3]

We define $\alpha(n) = \mu(\chi_n)$. Using the conditions on μ, it is direct to show

$$\mu(\alpha) \Rightarrow \exists n_1, n_2, n_3 \; [n_1 < n_2 < n_3 \;\wedge \chi_{n_1}(n_2) = \chi_{n_1}(n_3) = \chi_{n_2}(n_3) = 1] \tag{1}$$

and

$$\mu(1-\alpha) \Rightarrow \exists n_1, n_2, n_3 \; [n_1 < n_2 < n_3 \;\wedge \chi_{n_1}(n_2) = \chi_{n_1}(n_3) = \chi_{n_2}(n_3) = 0]. \tag{2}$$

Indeed, let us prove (1) (the other case is similar). If we have $\mu(\alpha)$, hence there exists n_1 such that $\alpha(n_1)$. By definition of α this means $\mu(\chi_{n_1})$ and so we have $\mu(\alpha\chi_{n_1})$. We thus know that there exists $n_2 > n_1$ such that $\alpha(n_2)\chi_{n_1}(n_2)$. This implies $\alpha(n_2) = \mu(\chi_{n_2})$, and hence $\mu(\alpha\chi_{n_1}\chi_{n_2})$. Hence there exists $n_3 > n_2$ such that $\alpha(n_3)\chi_{n_1}(n_3)\chi_{n_2}(n_3)$ which implies the proposition we want to establish: we get $\chi_{n_1}(n_2) = \chi_{n_2}(n_3) = \chi_{n_1}(n_3) = 1$.

Notice the apparent *impredicative* use of the ultrafilter μ in this argument; we did apply the functional μ to the function α, which is defined in term of μ.

Let us now interpret this proof in any Boolean measure algebra B. Let $x(n_1 n_2 n_3)$ be the Boolean which is the the truth value of $\chi_{n_1}(n_2) = \chi_{n_1}(n_3) = \chi_{n_2}(n_3)$ and let $\alpha \in B^N$ be defined by $\alpha(n) = \mu(\chi_n)$. We show

$$\bigvee_{n_1 < n_2 < n_3} x(n_1 n_2 n_3) = 1.$$

This follows from

$$\mu(\alpha) \leq \bigvee_{n_1 < n_2 < n_3} x(n_1 n_2 n_3)$$

[3] Of course, in this case, it is possible to prove directly ϕ in a combinatorial way. The goal of this section is only to illustrate a general technique on a simple example.

which corresponds to (1), and from

$$\mu(1-\alpha) \leq \bigvee_{n_1<n_2<n_3} x(n_1n_2n_3)$$

which corresponds to (2). Both assertions are proved like in the reasoning above using proposition 4.

In order to conclude from this to the *existence* of $n_1 < n_2 < n_3$ such that $x(n_1n_2n_3) = 1$ from this fact, we need a further result, which is proved in the next subsection.

2.3 Another Characterisation of C_0

The goal of this part is to show effectively that the initial Boolean measure algebra "reflects existential statements": if $b(n)$ is a sequence of elements in F_2 such that $\bigvee b(n) = 1$ in B then there exists n such that $b(n) = 1$. If $x \in \mathsf{C}$ and $n_1, \ldots, n_k$ is a sequence of natural number we write $x(n_1 \ldots n_k)$ for $x(n_1)(n_2)\ldots(n_k) \in \mathsf{C}$.

Lemma: If $x \in \mathsf{C}$ and $m_1 < m_2 < \ldots$ there exists k such that $x(m_1 \ldots m_k)$ is equal to 0 or 1.

Proof: This follows directly from proposition 1. □

Theorem 3: If $x \in \mathsf{C}_0$ and $n_1 < n_2 < \ldots$ there exists $i_1 < \ldots < i_p$ such that $x(n_{i_1} \ldots n_{i_p}) = 0$. [4]

Proof: We prove this by induction on the proof that $x \in \mathsf{C}_0$. If there exists N such that $xx(n) \in \mathsf{C}_0$ for all $n \geq N$, we can find i_0 such that $N \leq n_{i_0}$ because the sequence n_k is strictly increasing. Let $y = xx(n_1)$, by induction hypothesis there exists $i_1 < \ldots < i_p$ such that $i_0 < i_1$ and $y(n_{i_1} \ldots n_{i_p}) = 0$. Using the lemma, there exists k such that

$$z = x(n_{i_1} \ldots n_{i_p} n_{i_p+1} \ldots n_{i_p+k}) \quad \text{and} \quad z_0 = x(n_{i_0} n_{i_1} \ldots n_{i_p} n_{i_p+1} \ldots n_{i_p+k})$$

are equal to 0 or 1. Since

$$zz_0 = y(n_{i_1} \ldots n_{i_p})(n_{i_p+1} \ldots n_{i_p+k}) = 0$$

we have $z = 0$ or $z_0 = 0$. □

Corollary: If $x \in \mathsf{C}_0$ then there exists $i_1 < \ldots < i_p$ such that $x(i_1 \ldots i_p) = 0$.

Theorem 4: If $b \in F_2^N$ and $1 = \bigvee b(n)$ in B then there exists k such that $b(k) = 1$.

Proof: Let $x_k = 1 - b(k)$. We have $(x_0, x_0x_1, x_0x_1x_2, \ldots) \in \mathsf{C}_0$. By the corollary of theorem 3, this implies that there exists k such that $x_k = 0$, that is $b(k) = 1$. □

2.4 A Topos Theoretic Interpretation

Since the theory of sheaves and locales have not yet been developped in a predicative framework, in the sense of constructive type theory for instance [15][5], we limit ourselves to some informal connections with the usual notions of toposes of sheaves over a locale [3, 5].

Let B be any Boolean measure algebra. We can associate to B two locales such that, in the topos of sheaves over this locale, there exists a non principal ultrafilter defined over all internally decidable subsets of natural numbers. The first locale has for open the regular ideals of the Boolean algebra B, that is, ideals I such that $I^{\perp\perp} = I$ where $J^{\perp} = \{x \in \mathsf{B} \mid (\forall y \in J)xy = 0\}$. The other locale structure is suggested by the work [7] that applies to any Boolean σ-algebra: the open of this locale are the σ-complete ideals of B. In both cases, in the topos of sheaves over this locale, it can be checked that a Boolean function is, externally, an infinite sequence of elements of B. Also, the decidable propositions of this topos can be identified to the elements of B.

[4]Conversely, it can be shown using the law of excluded middle that if x is not in C_0, then there exists an infinite sequence $n_1 < n_2 < \ldots$ such that for all $i_1 < \ldots < i_p$, we have $x_{n_{i_1} \ldots n_{i_p}} \neq 0$. Thus, if we see the elements x of C as defining continuous Boolean functions $\phi(x) \in [\mathcal{N} \to F_2]$ over the Baire space, $x \in \mathsf{C}_0$ means, using the law of excluded middle, that for any sequence $n_1 < n_2 < \ldots$, we can find $i_1 < i_2 < \ldots$ such that $\phi(x)(n_{i_1} n_{i_2} \ldots) = 0$. With this interpretation, theorem 2 can be seen as an intuitionistic and algebraic formulation of the closed open Ramsey's theorem [8].

[5]See however [12, 7].

In the particular case where we start from the initial Boolean algebra B, this topos of sheaves satisfies furthermore a transfer principle for existential propositions. If a statement $\exists n\ \phi(n)$ where $\phi(n)$ is an external decidable proposition, is true in this topos, then this statement is true externally. This follows from theorem 4: if b_n is the Boolean value of $\phi(n)$, the internal truth of $\exists n\ \phi(n)$ means that 1 is the least upper bound of the sequence b_n in B. (Notice that this applies only if ϕ is *externally* decidable.)

Thus, this sheaf-model can be seen as a machine for producing constructive, combinatorial proofs of facts about natural numbers that can be expressed by an existential formula, and of which the standard proof is based on the existence of a non-principal ultrafilter over the natural numbers. This can be compared with the non-constructive use of Boolean cover in topos theory in the proof of Barr's theorem [3, 14].

Conclusion

We think that the result of theorem 2, that C_0 is an ideal, is interesting in itself. Notice that its proof is quite close to the proof of cut-elimination of infinitary propositional logic (see for instance [13]). On the other hand, as we already noticed, it can be seen as a reformulation of the classical clopen Ramsey's theorem [8], and provide a direct constructive proof of this result. We find it remarkable that this formulation comes from a problem that seems at first quite distinct, which was to understand construtively some use of a non-principal ultrafilter.

One question raised indirectly by this paper is what should be a formal description of non separable topological spaces. This question seems to apply as well to the description of Stone-Cech compactification described in [2]. This question applies in particular for a constructive interpretation of Hindman's theorem, which has a proof using non principal ultrafilters [9]. This theorem has a suggestive interpretation in terms of the Boolean algebra C. If $x \in \mathsf{C}$ let x^k denote the element $(x(k), x(k+1), \dots) \in \mathsf{C}$. Hindman's theorem can be interpreted as the result that the subset $H \subseteq \mathsf{C}$ defined inductively by the rules

$$\frac{}{0 \in H} \qquad \frac{\dots \quad xx(n)x^n \in H \quad \dots}{x \in H}$$

is an *ideal* of C. An effective proof of this result would be quite interesting.

This note is a further instance of the use of topological models in proof-theoretic problems (see [6] for another example). We hope to have shown that this can be seen as an illustration of Hilbert's program [11], reformulated in a constructive framework: non-principal ultrafilters are ideal objects, that can be eliminated, here by using suitable topological models, in any given proof of a concrete statement.

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