Uniform Kan filling

Let $\mathcal{C}$ the following category. The objects are finite sets $I, J, \ldots$. A morphism $\mathsf{Hom}(J, I)$ is a map $I \to \mathsf{dM}(J)$ where $\mathsf{dM}(J)$ is the free de Morgan algebra on J. The presheaf $\mathbb{I}$ is defined by $\mathbb{I}(J) = \mathsf{dM}(J)$. The presheaf $\mathbb{F}$ is defined by taking $\mathbb{F}(J)$ to be the free distributive lattice generated by formal elements $(j = 0)$, $(j = 1)$ for j in J, with the relations $(j = 0) \wedge (j = 1) = 0$.

If i is in I, we have maps (ib) in $\mathsf{Hom}(I - i, I)$ sending i to b, for $b = 0$ or 1. A *face* map is a composition of such maps. A *strict* map $\mathsf{Hom}(J, I)$ is a map $I \to \mathsf{dM}(J)$ which never takes the value 0 or 1. Any map f can be written uniquely $f = gh$ where g is a face map and h is strict.

The lattice $\mathbb{F}(I)$ has a greatest element < 1, the *boundary* element δ_I, which is the disjunction of all $(i = 0) \vee (i = 1)$ for i in I.

Using the canonical de Morgan algebra structure of $[0, 1]$, we can define a functor

$$\mathcal{C} \to \mathsf{Top},\ I \longmapsto [0, 1]^I$$

If u is in $[0, 1]^I$, think of u as an environment giving values in $[0, 1]$ to each i in I, so that iu in $[0, 1]$ if i in I. Any f in $\mathsf{Hom}(I, J)$ defines then $f : [0, 1]^I \to [0, 1]^J$ by $j(fu) = (jf)u$. If ψ is in $\mathbb{F}(I)$ and u in $[0, 1]^I$ then ψu is a truth value.

If $b = 0$ or 1 and i is in I, let $(ib) : [0, 1]^{I-i} \to [0, 1]^I$ be the map defined by by $i(ib)u = b$ and $j(ib)u = ju$ if $j \neq i$ in I.

We assume given a family of idempotent functions $r_I : [0, 1]^I \times [0, 1] \to [0, 1]^I \times [0, 1]$ such that

1. $r_I(u, z) = (u, z)$ iff $\delta_I u = 1$ or $z = 0$ and
2. for any *strict* f in $\mathsf{Hom}(I, J)$ we have $r_J(f \times \mathsf{id})r_I = r_J(f \times \mathsf{id})$

The last property can be reformulated as $r_I(u, z) = r_I(u', z') \to r_J(fu, z) = r_J(fu', z')$.
Such a family can for instance be defined as in [1] Figure 1.3 ("retraction from above center") [1].

Using this family, we can define for each ψ in $\mathbb{F}(I)$ an idempotent function

$$r_\psi : [0, 1]^I \times [0, 1] \to [0, 1]^I \times [0, 1]$$

having for fixed-points the element (u, z) such that $\psi u = 1$ or $z = 0$. This function r_ψ is completely characterized by the following properties

1. $r_\psi = \mathsf{id}$ if $\psi = 1$
2. $r_\psi = r_\psi r_I$ if $\psi \neq 1$
3. $r_\psi(u, z) = (u, z)$ if $z = 0$
4. $r_\psi((ib) \times \mathsf{id}) = ((ib) \times \mathsf{id})r_{\psi(ib)}$

For instance, these properties imply $r_{\delta_I}(u, z) = (u, z)$ if $\delta_I u = 1$ or $z = 0$ and so they imply $r_{\delta_I} = r_I$.

They also imply that $r_\psi(u, z) = (u, z)$ if $\psi u = 1$.

From these properties follows the uniformity of the family of functions r_ψ.

[1] Indeed, in this case, $r_I(u, z) = r_I(u', z')$ is equivalent to $(2 - z')(-1 + 2u) = (2 - z)(-1 + 2u')$, which implies $(2 - z')(-1 + 2fu) = (2 - z)(-1 + 2fu')$ if f is strict.

Theorem 0.1 *If f is in $\mathsf{Hom}(I,J)$ and ψ is in $\mathbb{F}(J)$ then $r_\psi(f\times\mathsf{id}) = (f\times\mathsf{id})r_{\psi f}$*

A particular case is $r_J(f\times\mathsf{id}) = (f\times\mathsf{id})r_{\delta_J f}$. Remark that, in general, $\delta_J f$ is not δ_I.

Proof. We prove this by induction on the number of element of I (the result being clear if I is empty). Using the last property (4) above, we can then assume that f is strict.

If $\psi f = 1$ then for any u in $[0,1]^I$ and z in $[0,1]$ we have $\psi f u = 1$ and so $r_\psi(fu,z) = (fu,z)$

If $\psi f \neq 1$ then we have $r_{\psi f} = r_{\psi f} r_I$ and $\psi \neq 1$, so $r_\psi = r_\psi r_J$. We thus have

$$(f\times\mathsf{id})r_{\psi f}(u,z) = (f\times\mathsf{id})r_{\psi f} r_I(u,z)$$

We write $(u',z') = r_I(u,z)$. We have $\delta_I u' = 1$ or $z' = 0$.

If $z' = 0$, then $r_J(f\times\mathsf{id})(u,z) = r_J(f\times\mathsf{id})r_I(u,z) = (fu',0)$ and so

$$r_\psi(f\times\mathsf{id})(u,z) = r_\psi r_J(f\times\mathsf{id})(u,z) = r_\psi(fu',0) = (fu',0) = (f\times\mathsf{id})r_{\psi f}(u,z)$$

If $\delta_I u' = 1$ then we can write $u' = (ib)v'$ for some i in I and v' in $[0,1]^{I-i}$. We then have

$$\begin{array}{rcll}
(f\times\mathsf{id})r_{\psi f}(u,z) & = & (f\times\mathsf{id})r_{\psi f}((ib)v',z') & \\
& = & (f\times\mathsf{id})((ib)\times\mathsf{id})r_{\psi f(ib)}(v',z') & \\
& = & (f(ib)\times\mathsf{id})r_{\psi f(ib)}(v',z') & \\
& = & r_\psi(f(ib)\times\mathsf{id})(v',z') & I.H. \\
& = & r_\psi r_J(f(ib)\times\mathsf{id})(v',z') & \psi\neq 1 \\
& = & r_\psi r_J(f\times\mathsf{id})((ib)v',z') & \\
& = & r_\psi r_J(f\times\mathsf{id})r_I(u,z) & \\
& = & r_\psi r_J(f\times\mathsf{id})(u,z) & \\
& = & r_\psi(f\times\mathsf{id})(u,z) & \psi\neq 1
\end{array}$$

So that $r_\psi(f\times\mathsf{id}) = (f\times\mathsf{id})r_{\psi f}$ as required. □

References

[1] R. Brown, P. J. Higgins and R. Sivera. *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids.* volume 15 of EMS Monographs in Mathematics , European Mathematical Society, 2011.