A cubical set model of type theory

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Univalent Foundation

We provide a constructive model of dependent type theory together with

$1_a : \mathsf{Eq}_A(a, a)$

$\mathsf{transp} : C(a) \to \mathsf{Eq}_A(a, x) \to C(x)$

$\mathsf{Eq}_{C(a)}(\mathsf{transp}\ u\ 1_a, u)$

$\mathsf{Eq}_{(\Sigma x:A)\mathsf{Eq}_A(a,x)}((a, 1_a), (x, p))$

Univalence Axiom ($+$ Propositional truncation $+$ Circle)

Dependent type theory: $\Pi, \Sigma, U, N, W(A, B), N_0, N_1, N_2 \ldots$

Computational Interpretation

Simple enough so that we have a Haskell implementation

j.w.w. C. Cohen, S. Huber and A. Mörtberg

In particular, we can effectively transport structures along equivalences

Construtive models

A *computational model* of univalence is the same as a *constructive model*

For dependent type theory, the computations are done in λ-calculus

For univalence, *nominal* extension of λ-calculus

Simplicial set model

A type is interpreted by a Kan simplicial set

A dependent type is interpreted by a Kan fibration $E \rightarrow A$

If there is a path between 0-elements a_0 a_1 of A then

the two fibers $E(a_0)$ and $E(a_1)$ should be equivalent

This is not provable intuitionistically (M. Bezem and T.C.)

Simplicial set model

The problem has to do with decidability of degeneracy

We need a simple framework to analyze this problem

Simple representation of homotopy types: Kan cubical sets (1955)

How to represent cubical sets?

Cubical sets

Nominal representation of cubical sets

$\{x, y, z, \dots\}$ the set of symbols/names/indeterminates

This set should be *discrete* and *infinite*

The symbols can be thought of as names of cartesian axes $x, y, z, \dots$

A Cubical set

Space P of polynomials (P. Aczel)

If k is a ring then $k[x, y, z, \dots]$ is a cubical set

A 0-element (a point) is an element of k

A 1-element (a line) is an element of $k[x]$ or $k[y]$ or $\dots$

A 2-element (a square) is an element of $k[x, y]$

A Cubical set

Consider $t(x, y)$ in $k[x, y, \dots]$

t represent a square with points $t(0, 0), t(0, 1), t(1, 0), t(1, 1)$

$t = rx + sy^2$ with r and s in k

That x *actually* appears in t may not be decidable

t *is independent of* x can be defined as $t = t_{(x=0)}$

A Cubical set

If $t(x,y) = u(y,z)$ then t and u depend only (at most) on y

$t(x,y) = t(0,y) = u(y,0) = u(y,z)$

A Cubical set

We have the *face* operations $t_{(x=0)}$ and $t_{(x=1)}$

Degeneracy operation: an element in $k[x]$ is also in $k[x,y]$

A line between two points p_0 and p_1 is a polynomial t such that

$$p_0 = t_{(x=0)} \qquad p_1 = t_{(x=1)}$$

Any two points are connected by a line

$$t = (1-x)p_0 + xp_1$$

Path space

A *point* of Path(P) is of the form $\langle x\rangle t$ with t in $k[x]$

$\langle x\rangle t = \langle y\rangle u$ if and only if $t = u_{(y=x)}$

A *line* is of the form $\langle x\rangle v$ with v in $k[x, y]$, and so on

Path(P) is also a cubical set

If $w = \langle x\rangle(x + y^2)$ then $w_{(y=0)}$ is $\langle x\rangle x$

$\langle x\rangle v$ is the operation of *name abstraction*

Path

If p_0 and p_1 are polynomials we define $\mathsf{Eq}_{\mathsf{P}}(p_0, p_1)$

An element of this cubical set is an element $\langle x \rangle t$ in $\mathsf{Path}(\mathsf{P})$ such that

$p_0 = t_{(x=0)} \qquad\qquad p_1 = t_{(x=1)}$

$\mathsf{Eq}_{\mathsf{P}}(p_0, p_1)$ is also a cubical set

Cubical sets

Definition: *A cubical set is a set valued functor on the category of finite sets and morphims* $f : I \to J$ *given by* $I = I_0, I_1, I'$ *and injection* $I' \to J$

An element of $A(x, y)$ is of the form $t(x, y)$

If $I \subseteq J$ the corresponding map $A(I) \to A(J)$ is an injection

We want to consider $t(x, y)$ to be also in $A(x, y, z)$

$$t(x, y) = t(x, y, z)$$

Cubical set

Given such a cubical set A we associate the set of pairs (I, u) with u in $A(I)$

$(I, u) \simeq (J, v)$ iff $u_{|I \subseteq I \cup J} = v_{|J \subseteq I \cup J}$ in $A(I \cup J)$

Any element depends on at most finitely many symbols

We can rename and have $0, 1$ substitutions

In this way we define a nominal set

Polynomials

$A(x, y)$ set of polynomials using at most x and y

In this case we have $A(I) \subseteq A(J)$ if $I \subseteq J$

The associated nominal set is $\mathsf{P} = k[x, y, \ldots]$

Cubical set/Nominal set

Similar categories have already been considered

Staton (2010) and more recently P.A. Mellies

A restricted to finite sets and injective maps preserve pull-backs

A cubical set can be seen as a nominal set with $0, 1$-substitutions

Cf. "An equivalent presentation of the Bezem-Coquand-Huber category of cubical sets", A. M. Pitts. Preprint arXiv, December 2013.

Cubical sets

Any topological space S defines a cubical set

$S(x, y)$ to be the set of continuous functions $[0, 1]^{\{x,y\}} \rightarrow S$

Cubical set

If A is a cubical set we can define a new cubical set $\mathsf{Path}(A)$

A 0-object in $\mathsf{Path}(A)$ should be of the form $\langle x\rangle t$ for some t in $A(x)$

$\mathsf{Path}(A)(x,y)$ has elements $\langle z\rangle t(x,y,z)$ for some t in $A(x,y,z)$

Cubical set

Unit interval $\mathbf{I}$ as a cubical set

$\mathbf{I}(x, y)$ is the set $\{0, 1, x, y\}$

There is a separated product $B * \mathbf{I}$

An element is a pair b, x where b is independent of x

$\mathsf{Path}(A)$ is an exponential for this product, right adjoint to separated product

If $u(x, y) = \langle z \rangle v(x, y, z)$ we cannot apply u to x or to y

We can apply u to z or to 0 or to 1

Map on Paths

If $\sigma : A \to A$ then we can define

$\mathsf{mapOnPaths}(\sigma) : \mathsf{Path}(A) \to \mathsf{Path}(A)$

$\mathsf{mapOnPaths}(\sigma)\ (\langle x \rangle t) = \langle x \rangle (\sigma\ t)$

This holds *as a definition*

What is a model of type theory

Type theory is a generalized algebraic theory (Cartmell)

Sorts, *operations* and *equations*

Each sort is interpreted as a *set*

Interpretation of each operation

Check that the required equations are valid

Like for equational theories

What is a model of type theory

The sorts are

$\Gamma \vdash$ *context*

$\sigma : \Delta \to \Gamma$ *substitution*

$\Gamma \vdash A$ *type*

$\Gamma \vdash t : A$ *element*

Some operations $\Gamma.A, \quad A\sigma, \quad t\sigma, \quad (\sigma, u), \quad \mathsf{app}(v, u), \ldots$

Some equations $\mathsf{q}(\sigma, u) = u, \quad \mathsf{p}(\sigma, u) = \sigma, \quad , \ldots$

Cubical set model

A *context* is interpreted by a cubical set

A *substitution* by a natural transformation

A *type* $\Gamma \vdash A$ by a family of sets with restriction operations

If ρ is in $\Gamma(I)$ then $A\rho$ is a set

If u in $A\rho$ then uf is in $A(\rho f)$

Cubical set model

If ρ is an element in $\Gamma(x, y)$ then $A\rho$ is a set of squares

If ρ in $\Gamma(x, y)$ and u in $A\rho$ we can form $u_{(x=0)}$ in $A\rho_{(x=0)}$

An *element* $\Gamma \vdash a : A$ is interpreted by a family $a\rho$ in $A\rho$ such that

$$(a\rho)f = a(\rho f)$$

Cubical set model

For instance if T is a cubical set we have

$$p : T, q : T \vdash \mathsf{Eq}_T(p, q)$$

Γ is $p : T, q : T$

A is $\mathsf{Eq}_T(p, q)$

If u and v in $T(x, y)$ then $(p = u, q = v)$ defines ρ in $\Gamma(x, y)$

$A\rho$ is set of lines joining u and v depending on x and y

If l is such a line, we can consider $l_{(y=0)}$ joining $u_{(y=0)}$ and $v_{(y=0)}$

Cubical set model

Let P be the cubical set of polynomials

We have $\alpha : (\Pi\ p_0\ p_1 : \mathsf{P})\mathsf{Eq}_{\mathsf{P}}(p_0, p_1)$

$\alpha\ p_0\ p_1 = \langle x \rangle t(x)$

$t(x) = (1 - x)p_0 + xp_1$

If p_0 and p_1 depend at most on y, z then

$(\alpha\ p_0(y, z)\ p_1(y, z))_{(y=0)} = \alpha\ p_0(0, z)\ p_1(0, z)$

Uniformity condition

Contexts and Types

A *context* is interpreted by a cubical set

A *type* $\Gamma \vdash A$ is interpreted by a "Kan fibration"

Contexts and Types

Kan structure: if we have α in $\Gamma(x)$ there should be a map

$$A\alpha_0 \to A\alpha_1$$

$$u \longmapsto A\alpha^+ u$$

We also have a line $A\alpha \uparrow u$ in $A\alpha$ connecting u and $A\alpha^+ u$

In general, any open box can be filled

Uniformity condition

These are *operations*

u is independent of x

$A\alpha^{+}u = A\alpha^{+}_{(x=y)}u$ if y is fresh

$(A\alpha \uparrow u)_{(x=0)} = u \qquad (A\alpha \uparrow u)_{(x=1)} = A\alpha^{+}u$

We allow u to "depend" on more symbols $y, z, \ldots$

$(A\alpha \uparrow u)_{(y=0)} = A\alpha \uparrow u_{(y=0)}$

This strengthens the Kan condition

This solves the problem with equivalence of fibers

What is a Kan Fibration

This generalizes and refines the notion of family of sets in Bishop's framework

Cf. Exercice 3.2 in Bishop's book

In the first edition, only families over discrete sets are considered

The next edition presents a more general definition, due to F. Richman

Uniformity condition

Polynomial example

Any category defines a cubical set

It satisfies the uniform Kan condition if the category is a groupoid

Model of type theory

Theorem: *If* $\Gamma \vdash A$ *and* $\Gamma.A \vdash B$ *have a Kan structure then so does* $\Gamma \vdash \Pi\ A\ B$

Theorem: *If* $\Gamma \vdash A$ *has a Kan structure and* $\Gamma \vdash a_0 : A$ *and* $\Gamma \vdash a_1 : A$ *then*

$$\Gamma \vdash \mathsf{Eq}_A\ a_0\ a_1$$

has a Kan structure

Universe

U defined as a cubical set

Any finite set of symbols I defines (Yoneda) a cubical set

An element of $\mathsf{U}(I)$ is given by a small type $I \vdash A$ with a Kan structure

This defines U as a *cubical set*

$\mathsf{U}(x)$ set of $x \vdash A$

line connecting $A_{(x=0)}$ and $A_{(x=1)}$ satisfying the Kan condition

Univalence

What is an equivalence $\sigma : A \to B$?

We can transform any equivalence to an element E in $\mathsf{U}(x)$

$E_{(x=0)}$ is A and $E_{(x=1)}$ is B

A line $a \to b$ in E is a line $\sigma a \to b$ in B

Univalence

This gives a map

$(\Pi A\ B : U)\ \mathsf{Equiv}(A, B) \rightarrow \mathsf{Eq}_U(A, B)$

It is an inverse of the canonical map

$(\Pi A\ B : U)\ \mathsf{Eq}_U(A, B) \rightarrow \mathsf{Equiv}(A, B)$

and hence this map is an equivalence (this is the Axiom of Univalence)

Theorem: *The cubical set model satisfies the Axiom of Univalence*

Kan filling for the universe

Natural composition operation (like composition of relation)

Theorem: (G. Gonthier, S. Huber) *The universe has a Kan structure*

Implementation

With C. Cohen, S. Huber, A. Mörtberg

We “force” the syntax to have univalence

Kan filling operations are defined by induction on the types

Purely operational description of the model with U : U

The implementation can be described as a generalized algebraic theory

To be checked: subject reduction, which is purely syntactical

Implementation

We have added dependent equality

$\mathsf{Eq}_p(a, b)$ if $p : \mathsf{Eq}_U(A, B)$ and $a : A$ and $b : B$

$\mathsf{Eq}_{1_A}(a_0, a_1)$ is convertible to $\mathsf{Eq}_A(a_0, a_1)$

Simpler equality for sigma types

Better definitional equalities for the "map on Paths" operation

$\mathsf{mapOnPaths(id)}(\langle x\rangle u) = \langle x\rangle u$

Implementation

Realizability semantics in a nominal calculus

I don't expect any problem to prove normalization with a hierarchy of universes

We can add resizing rules for propositions

Do we still have normalization if we add these rules?

Implementation

With resizing rules we have a small *set* Ω

$\Omega = (\Sigma X : U)\mathsf{prop}(X)$

The sets form a topos

We have a computational interpretation of unique choice

Examples

-Computing with equivalence classes (variation of V. Voevodsky's example)

-non trivial equality of N_2 with itself

-N and $N + 1$ are isomorphic

Hence we can transport any structure on N to a structure on $N + 1$

Structure with vectors on N

-$(\Pi X\ Y : U)(X \times Y \rightarrow X)$ and $(\Pi X\ Y : U)(X \rightarrow Y \rightarrow X)$ are equal

Propositional truncation

If A Kan cubical set we define $\mathsf{inh}(A)$

We add operations $\alpha_x\ p\ q$ for $p\ q$ in A independent of x

We add Kan filling operations

These operations are added as *constructors*

Circle

Similarly we can define the circle S^1

We need a diagonal operation for the elimination rule

Diagonal

What we need is an operation of *total concretion*

Cf. *Names and Symmetry in Computer Science* A.M. Pitts, 9.7

We want to extend the operation $\mathsf{Path}(A) * \mathbf{I} \to A$ to a total operation

We can do it up to propositional equality

Hence we get an elimination rule where the object equality is definitional

but the path equality is only propositional

Can we do it in a definitional way?

Can we modify the model to allow a diagonal operation?

Variation: parametricity

We force an equivalence to be a line

We can instead force an arbitrary relation to be a line

We get a model with internal parametricity

We also an implementation of this variation

C.f. work of J.P. Bernardy and G. Moulin

Extension of type theory

These are two example of presheaf models

Adding “ideal” elements

These presheaf models are described by a generalized algebraic theory

We still have a notion of definitional equality

Some references

"A model of type theory in cubical sets"
M. Bezem, T.C. and S. Huber, to appear in the proceeding of TYPES 2013

Names and Symmetry in Computer Science
A.M. Pitts

"An equivalent presentation of the Bezem-Coquand-Huber category of cubical sets"
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Some references

The Univalent Foundation Program
Homotopy Type Theory: Univalent foundation of mathematics

V. Voevodsky Univalent foundation home page and
"Experimental library of univalent foundation of mathematics"