A SYNTACTICAL PROOF OF THE MARRIAGE LEMMA

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Abstract. We give a proof of the classical Marriage Lemma [5] using completeness of hyperresolution. This argument is purely syntactical, and extends directly to the infinite case. As an application we give a purely syntactical version of a proof that resolution is exponential on the pigeon-hole principle.

Introduction

The resolution rule [10] can be stated without references to logical connectives. It manipulates clauses, that can be seen abstractly as a pair of finite sets of "token" or atomic propositions. Despite, or maybe because of, this simplicity, it has deep connections with various parts of mathematics. Some of these connections were pointed out early on in [7, 6], respectively in the field of discrete mathematics and algebra. In [2, 3] other connections are described in the framework of entailment relations [11], an abstract version of Gentzen multi-conclusion sequent calculus, which can be seen as a variation on the resolution calculus. In [4] the connections with resolution and hyperresolution [10] are analysed in detail. In resolution calculus, we are interested in the consequences of a theory, that is a collection of given clauses $X_i \vdash Y_i$. Here we write $X, Y, Z, \ldots$ for finite sets of token, and we write

$$x_1, \ldots, x_n \vdash y_1, \ldots, y_m$$

the clause $\neg x_1, \ldots, \neg x_n, y_1, \ldots, y_m$. In many mathematical examples, one can give a direct, non inductive, description that captures exactly the consequences of a given theory.

As a typical example, if S is a commutative group, the consequences of the theory

- $x, -x \vdash,$
- $x + y \vdash x, y$

have the following direct description: $X \vdash Y$ iff a non empty sum of elements in X is equal to a sum of elements in Y. The *models* of this theory are in canonical correspondance with the linear preorderings of S and the intuitive meaning of the atomic proposition x is the statement $x > 0$. [1] This example shows how a mathematical notion can be captured and analysed by a simple propositional theory.

In this note, we give an example in the field of combinatorics, and present a purely syntactical proof of the classical Marriage Lemma [5] also known as Hall's theorem. Though the core of the argument is then very close to the one of Rado [9], we feel that there is some interest in connecting it to hyperresolution, and the present note can be seen as yet another example of the use of hyperresolution in combinatorics [7]. We apply then this to describe a syntactical variation of the lower bound proof for the Pigeon-Hole principle presented in [1].

[1] The characterisation of $\vdash$ follows easily from the remark that the rule $x, y \vdash x + y$ can be derived by resolution from $x \vdash x + y, -y$ and $y, -y \vdash$.

1. Hyperresolution

To simplify our arguments we use the completeness of hyperresolution. This is used also in [7, 6]. To a sequent $X_i \vdash Y_i$ we associate the corresponding hyperresolution rule

$$\frac{a_1, Y \ \dots \ a_n, Y}{Y} \qquad \textit{provided } X_i = a_1, \dots, a_n \textit{ and } Y_i \subseteq Y$$

It can then be shown [10] that $\vdash Y$ is a (semantical) consequence of a set of sequents $X_i \vdash Y_i$ iff Y can be derived using the hyperresolution rules associated to these sequents.

One can give also a syntactical version of this result, which is the form we shall use. We consider the set of rules

$$a \vdash a$$

$$\frac{X' \supseteq X \qquad X \vdash Y \qquad Y \subseteq Y'}{X' \vdash Y'}$$

$$\frac{X \vdash Y, a \qquad a, X \vdash Y}{X \vdash Y}$$

then a sequent $X \vdash Y$ is a consequence of a set of sequents $X_i \vdash Y_i$ using these rules iff Y is a consequence of the sequents $X_i \vdash Y_i$ and the sequents $\vdash x$, for $x \in X$ using hyperresolution. This is a way to express completness of hyperresolution w.r.t. resolution [10].

2. Pigeon-Hole Theory

We suppose given two sets finite sets P and H. We take $S = P \times H$. The rules are

$$(a, h_1), (a, h_2) \vdash \qquad T_1$$

meaning that a pigeon cannot be in two different holes, and

$$(a_1, h), (a_2, h) \vdash \qquad T_2$$

meaning that there is at most one pigeon in one hole, and finally

$$\vdash \{(a, h) \mid h \in H\} \qquad C$$

meaning that each pigeon a is in at least one hole. We let T be the set of all these rules ("pigeon-hole theory").

The theory T is the union of the three theories T_1, T_2 and C.

We give a direct characterisation of consequences of T. Let a *large rectangle* be a set of the form $P_1 \times H_1$ with $P_1 \subseteq P,\ H_1 \subseteq H$ and $|H| < |P_1| + |H_1|$.

Theorem 2.1. *$Y \subseteq P \times H$ is a consequence of T iff it is a consequence of T_1, C iff it contains a large rectangle.*

A simple corollary is that the theory T_1, C is inconsistent iff $|H| < |P|$ since in this case we can take $P_1 = P$ and $H_1 = \emptyset$.

Proof. It is clear that any subset containing a large rectangle is a consequence of the theory T, or even the theory T_1, C[2].

[2]Notice that non syntactically, a model of T is exactly a one-to-one map $f : P \to H$. If $|H| < |P_1| + |H_1|$ the image of P_1 by f has to meet H_1. Hence, any model of T is a model of $P_1 \times H_1$. This non syntactical argument can be replaced by a direct hyperresolution derivation, built by induction on the size of P_1.

We show that the set of such subsets of $P \times H$ is closed by the hyperresolution rule. We have then three cases to consider.

Y is derived from $Y,(a,h_1)$ and $Y,(a,h_2)$ and we can assume $(a,h_1) \notin Y,\ (a,h_2) \notin Y$. We have by hypothesis that $Y,(a,h_1)$ and $Y,(a,h_2)$ both contain a large rectangle. We can assume that these large rectangle contain respectively $(a,h_1),\ (a,h_2)$, otherwise this large rectangle will be contained in Y and we have finished. Thus, we get P_1, H_1, P_2, H_2 such that

$$|H| < |P_i \cup \{a\}| + |H_i \cup \{h_i\}| = 2 + |P_i| + |H_i| \qquad (*)$$

and the large rectangle

$$(P_i \cup \{a\}) \ \times (H_i \cup \{h_i\})$$

is a subset of $Y,(a,h_i)$. Since $(a,h) \in Y$ if $h \in H_2$ and (a,h_1) is not in Y it follows that h_1 is *not* in H_2. Similarly, h_2 is not in H_1. Also, the two products

$$(P_1 \cap P_2) \ \times \ (H_1 \cup H_2 \cup \{h_1, h_2\}) \qquad\qquad (P_1 \cup P_2 \cup \{a\}) \ \times \ (H_1 \cap H_2)$$

are subsets of Y.

From $(*)$ we derive

$$2|H| < 3 + |P_1| + |H_1| + |P_2| + |H_2|$$

but this last expression can be rewritten to

$$|P_1 \cap P_2| + |H_1 \cup H_2 \cup \{h_1, h_2\}| + |P_1 \cup P_2 \cup \{a\}| + |H_1 \cap H_2|$$

and hence we have either

$$|H| < |P_1 \cap P_2| + |H_1 \cup H_2 \cup \{h_1, h_2\}|$$

or

$$|H| < |P_1 \cup P_2 \cup \{a\}| + |H_1 \cap H_2|;$$

hence the result.

The case where Y is derived from $Y,(a_1,h)$ and $Y,(a,h_2)$ is similar.

The last case is when Y contains $\{(a,h) \mid h \in H\}$. We can take $P_1 = a$ and $H_1 = H$. □

The core of the argument is quite similar to the one in [9] (but was rediscovered independently). Thus the contribution of the present note can be seen as explaining how to connect Rado's argument with hyperresolution.

The next result can be stated without assuming H or P finite. It is a purely syntactical version of the classical Marriage Lemma [5]. Notice that we get directly the infinite case as well, without any form of the axiom of choice.

Corollary 2.2. *Suppose given $A_a \subseteq^{\text{fin}} H$ for each $a \in P$ and consider the theory*

$$\vdash \{(a,h) \mid h \in A_a\} \qquad D$$

that expresses that each pigeon a has to be in a hole in A_a. Then the theory T_1, T_2, D is contradictory iff the theory T_1, D is contradictory iff there exists $P_1 \subseteq^{\text{fin}} P$ such that $|\cup_{a \in P_1} A_a| < |P_1|$.

Proof. If the theory T_1, T_2, D is contradictory, there exists a finite subset from which we can deduce a contradiction. This is a purely syntactical observation: a proof of a contradiction is a finite object, which involves only a finite subset of the set of axioms. In this subset, only a finite number of pigeons and holes $P_0 \subseteq P,\ H_0 \subseteq H$

are mentionned, and, by enlarging H_0 if necessary, we can assume that $A_a \subseteq H_0$ for all $a \in P_0$. We consider then the theories

$$(a, h_1), (a, h_2) \vdash \qquad T_1'$$

for each $a \in P_0$, and $h_1 \neq h_2 \in H_0$ and

$$(a_1, h), (a_2, h) \vdash \qquad T_2'$$

for each $a_1 \neq a_2 \in P_0$, and $h \in H_0$ and

$$\vdash \{(a, h) \mid h \in H_0\} \qquad C'$$

for each $a \in P_0$, and

$$\vdash \{(a, h) \mid h \in A_a\} \qquad D'$$

for each $a \in P_0$. We know that T_1', T_2', D' is contradictory. This implies that the clause

$$\{(a, h) \mid a \in P_0,\ h \in H_0 - A_a\}$$

is a consequence of the theory T_1', T_2', C'. The result follows then from the theorem: we have P_1, H_1 such that

$$P_1 \times H_1 \subseteq \{(a, h) \mid a \in P_0,\ h \in H_0 - A_a\}$$

such that

$$|H_0| < |P_1| + |H_1|$$

We have then $H_1 \subseteq H_0 - \cup_{a \in P_1} A_a$, and hence

$$|H_1| \leq |H_0| - |\cup_{a \in P_1} A_a|$$

which implies

$$|\cup_{a \in P_1} A_a| < |P_1|$$

as desired. □

In *non syntactical* terms, it means that if we assume $|P_1| \leq |\cup_{a \in P_1} A_a|$ for any finite subsets P_1 of P then there exists an injective map f in $(\Pi a \in P) A_a$. Indeed, the theory T_1, T_2, D is then non contradictory, and hence has a model, which is such a function f. This is the usual formulation of the Marriage Lemma [5, 9]. To get this version from the corollary, it is enough to use compactness (in its model theoretic form) of propositional logic.

Let us give another application, characterising exactly all the consequences in the case where H, P are finite.

Corollary 2.3. *Any clause consequence of T_1, T_2, C, and not of T_1, T_2 has the form*

$$(a_1, h_1), \ldots, (a_n, h_n) \vdash Y$$

where $a_i \longmapsto h_i$ is a partial one-to-one map, and $P_1 \times H_1 \subseteq Y$, where P_1 is disjoint from $a_1, \ldots, a_n$, and H_1 is disjoint from $h_1, \ldots, h_n$ and $|H| < |P_1| + |H_1| + n$.

Proof. Since the clause is not a consequence of T_1, T_2 it has the form

$$(a_1, h_1), \ldots, (a_n, h_n) \vdash Y$$

where $a_i \longmapsto h_i$ is a partial one-to-one map. Then $\vdash Y$ is a consequence of $T_1, T_2, C, (a_1, h_1), \ldots, (a_n, h_n)$, hence the result from the theorem. □

From this result, we get an explicit description of the *prime implicants* of the theory T_1, T_2, C [8].

3. Application to the analysis of the Pigeon-Hole Principle

We present a variation on the simplified version of the lower bound proof of [1]. We consider the theory T_1, T_2, C in the case where P, H are finite, $|H| = N = |P|-1$ and we want to show that any proof of the empty clause by resolution has to be exponential.

Our argument avoids the non syntactical notion of critical assignement [1]. We say that a clause C is *killed* by a token (a,h) iff C is a consequence of the theory $T_1, T_2, (a,h)$. A clause is *large* iff it is killed by at least $2/9$th of the atoms. We limit ourselves to show that in any proof by resolution appears a large clause. In [1], it is shown how the exponential lower bound result follows from this fact by using the method of random restrictions.

Proposition 3.1. *Any resolution proof of the empty clause from T_1, T_2, C contains a large clause.*

Proof. It is easily seen that there is one clause in this derivation that can be derived from less than $2/3$ of the clauses C but cannot be derived from less than $1/3$ of these clauses. By the second corollary of the theorem, this clause has the form

$$(a_1, h_1), \dots, (a_n, h_n) \vdash Y$$

where $a_i \longmapsto h_i$ is a partial one-to-one map, and $P_1 \times H_1 \subseteq Y$, where P_1 is disjoint from $a_1, \dots, a_n$, and H_1 is disjoint from $h_1, \dots, h_n$ and $N < |P_1| + |H_1| + n$. If $k = |P_1|$ we have by assumption $1/3(N+1) \leq k \leq 2/3(N+1)$. This clause is killed by all (a_i, h), $h \neq h_i$ and all elements of $P_1 \times H_1$. Hence, it is killed by at least

$$n(N-1) + k(N+1-n-k) \geq k(N+1-k) \geq 2/9(N+1)^2$$

atoms. □

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