# A SIMPLE PROOF OF STONE-WEIRSTRASS

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Abstract. We give a simple proof of Stone-Weierstrass theorem. This argument shows how to compute arbitrary approximation of the square root of an element in $[0,1]$ in any ordered ring with an element $1/2$.

## Introduction

We give a constructive direct proof of the key lemma for showing Stone-Weirstrass theorem [1]. This argument shows how to compute arbitrary approximation of the square root of an element in $[0,1]$ in any ordered ring with an element $1/2$.

## Ordered Rings and Square Roots

Let $R$ be a commutative ring with a *cone* $P$, that is a subset containing 0 and closed by addition and multiplication. We assume also that there exists an element $1/2 \in P$ such that $2.1/2 = 1$. We write also $x \leq y$ for $y - x \in P$, and $a \leq x \leq b$ for $a \leq x$ and $x \leq b$ and $x \in [0,1]$ for $0 \leq x \leq 1$. Given $x \in [0,1]$ we show how to compute arbitrary approximation of the square root of $x$. More precisely given $N$ we show how to compute $y \in [0,1]$ such that

$$0 \leq y^2 - x \leq 1/2^N$$

For this we consider the two sequences $(y_n)$ and $(z_n)$ of elements in $[0,1]$ defined by $y_0 = z_0 = 0$ and

$$y_{n+1} = 1/2(x + y_n^2) \qquad z_{n+1} = 1/2(1 + z_n^2)$$

Clearly, we have $y_n \leq z_n$ for all $n$

I claim that we have for all $n$

$$y_n \leq y_{n+1} \quad z_n \leq z_{n+1} \quad y_{n+1} - y_n \leq z_{n+1} - z_n$$

This is proved by induction from the equalities

$$y_{n+1} - y_n = 1/2(y_n + y_{n-1})(y_n - y_{n-1}) \quad z_{n+1} - z_n = 1/2(z_n + z_{n-1})(z_n - z_{n-1})$$

It follows that we have

$$(1 - y_n)^2 - (1 - x) = 2(y_{n+1} - y_n) \leq 2(z_{n+1} - z_n)$$

In order to conclude, all is left is to show that $(z_n)$ has for limit 1. We know that $0 \leq z_n \leq z_{n+1} \leq 1$ and we have

$$1 - z_{n+1} = (1 - z_n)1/2(1 + z_n) \leq (1 - z_n)(1 - \epsilon/2)$$

if $z_n \leq 1 - \epsilon$. This shows that if $(1 - \epsilon/2)^N \leq \epsilon$ we have $1 - z_n \leq \epsilon$ for all $n \geq N$.

## References

[1] Stone, M. H. The generalized Weierstrass approximation theorem. Math. Mag. 21, (1948). 167–184, 237–254.

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