About Stone's notion of Spectrum

Thierry Coquand

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Introduction

Stone duality between Boolean algebra and compact totally disconnected spaces provides an algebraic and point-free presentation of a large class of topological spaces. A generalisation of this presentation, also due to Stone, is described in [Johnstone, Stone]: the class of spaces is larger, we get all compact Hausdorff spaces, and the representation is still algebraic, using divisible archimedean rings. These rings are now torsion free, contrary to the case of Boolean rings. Actually, these rings appear implicitely in analysing problems of measure on Boolean algebras [Tarski]. We give here a variation of the treatment of [Johnstone, Stone] [1] which can be seen also as a constructive "real" version of Gelfand duality in the style of [BM]. Our main result is a localic proof of the fact that the uniform norm of the Gelfand transform of an element is equal to its norm. A related result is a purely localic formulation of the theorem of Kadison-Dubois [BS].

1 Preordered Ring

We start from a ring R with a preorder $\leq$ so that $0 \leq a^2$ for all $a \in R$. To give such a preorder is equivalent to give a *cone*: a subset P which contains all squares and is closed by addition and multiplication; we can then define $a \leq b$ as $b - a \in P$ [Krivine]. In general, a P-*cone* is a subset closed by addition, multiplication and containing P. The set P of elements $a \geq 0$ is clearly the least P-cone. If Π is a P-cone, the P-cone generated by Π and an element $a \in R$ is the set $\Pi + a\Pi$ since P and hence Π contains all the squares.

We suppose now the ring R to be *divisible*: for each $n \geq 1$ there exists b such that $n.b = 1$, and *archimedian*: for any $a \in R$ there exists $r \in Q$ such that $a \leq r$. Notice that we can consider R as a Q-algebra. Notice also we have $1/n \in P$ for any $n > 0$, which follows from the fact that $1/n$ for $n > 0$ can be written as a sum of squares. By the same remark, if $0 \leq x$ in R and $r > 0$ in Q we have $0 \leq rx$.

We can consider R as a vector space on Q, because R is divisible, and we have a semi-normed structure on R [BM, MP] by defining $a \in N(q)$ as $a \leq q' \wedge -a \leq q'$ for some $q' < q$. We can verify that for all $q, q' > 0$

- if $x \in N(q)$ then there exists $q' < q$ such that $x \in N(q)$; simply because the rationals are dense

[1] Our notion of constructivity is a little stronger than the one used in the framework of topos theory and the reader can check that the power set axiom is never used. We think our arguments are valid in the sense of the *geometric logic* developped by S. Vickers [Vickers].

- for any x there exists q such that $x \in N(q)$; because the ring is supposed to be archimedian
- if $x \in N(q),\ x' \in N(q')$ then $x + x' \in N(q + q')$; because the set P is closed by addition
- if $x \in N(q)$ and $r > 0$ then $xr \in N(qr)$; because the set P is closed by multiplication
- if $x \in N(q)$ then $-x \in N(q)$; direct by definition
- $0 \in N(q)$; because we can find q' such that $0 < q' < q$

2 Representation of Compact Regular Spaces

For the following theorem, we assume that R is divisible and archimedian, and that all rational $r \geq 0$ are in P but we do not assume that P contains all squares.

Theorem 2.1 *The following geometrical propositional theory [MP], with atoms $a, b, \ldots$ all elements of R*

1. $a, -a \vdash$
2. $a + b \vdash a, b$
3. $a \vdash$ *if* $a \leq 0$ *(i.e.* $-a \in P$*)*
4. $\vdash 1$
5. $a, b \vdash ab$
6. $ab \vdash a, -b$
7. $a \vdash \bigvee_{r>0} \quad a - r$

describes a compact regular space: the points of the spaces are the models of this propositional theory, and each $a \in R$ defines the basic open of models satisfying the proposition a. A point of this space defines exactly one ring morphism $\phi : R \to \mathsf{R}$ into the Dedekind reals such that $a \geq 0$ implies $\phi(a) \geq 0$.

In the theory above a stands intuititively for $a > 0$. Furthermore we have adopted the following convention of representing both conjunction and disjunction simply by the symbol ",". It means conjunction if used at the left of $\vdash$ and disjunction if used at the right of $\vdash$. This convention comes from Gentzen's sequent calculus [CC00, Sco74].

Proof. We start by noticing some direct consequences of this axiomatisation: first $a, b \vdash a + b$ follows from the axioms $1, 2$. Indeed we have $a = a + b + (-b)$ and hence $a \vdash a + b, -b$ by 2. Since $b, -b \vdash$ by 1 it follows that we have $a, b \vdash a + b$. Second we have $a \vdash a + b$ if $b \in P$. We have seen that $a \vdash a + b, -b$ and hence by 3 it follows that we have $a \vdash a + b$ if $b \in P$. We notice also that we can deduce $\vdash r$ for and $r > 0$ in Q; this follows from (and actually is equivalent to) the axiom 4.

We follow then the method presented in [BM]. We let $Sp_{fin}(A)$ be the space described by all finitary axioms (all except the last axiom), and we write $X \vdash_{fin} Y$ if $X \vdash Y$ follows from these finatary axioms. We show that we can interpret all the axioms of $Sp(A)$ by interpreting a as $\bigvee_{r>0} a - r$ in $Sp_{fin}(A)$. The compactness of $Sp(A)$ follows then directly, as explained in [BM]. Notice that the theory $Sp_{fin}(A)$ is a possible axiomatisation of the notion of total preordering of the ring R [Johnstone]; we can then check

1. for any $r > 0, s > 0$ we have $a - r, -a - s \vdash$ because $(a - r) + (-a - s) = -r - s < 0$
2. for any $r > 0$ we have $a+b-r \vdash a-r/2, b-r/2$ and hence $a+b-r \vdash \bigvee_{s>0} a-s, \bigvee_{t>0} b-t$ in $Sp_{fin}(A)$
3. if $r > 0$ and $a \leq 0$ then $a - r \vdash$ because $a - r \leq 0$
4. we have $\vdash s$ for any rational $s > 0$ hence $\vdash \bigvee_{r>0} 1 - r$ holds
5. we have, if $r, s > 0$, that $a - r, b - s \vdash ab - rs$ because $ab - rs = r(b - s) + s(a - r) + (a - r)(b - s)$ and hence $a - r, b - s \vdash \bigvee_{t>0} ab - t$
6. we need the archimedian property: take $M, N > 0$ such that $a, -a \leq M$ and $b, -b \leq N$. We have $ab - r \vdash a - r_1/N, -b - r_2/M$ as soon as $r_1 + r_2 + r_1 r_2/MN \leq r$ since then $ab - r \leq (a - r_1/N)(b + r_2/M)$
7. if $r > 0$ we have $a - r \vdash \bigvee_{s>0} \bigvee_{t>0} \; a - s - t$ because we have $a - r \vdash a - s - t$ if $s = t = r/2$

Next we notice that we have $\vdash a - r, s - a$ if $r < s$ and $a - r, r - a \vdash$. This shows that the open set defined by $a - r$ (intuitively, the set point of points for which $a > r$) is well-inside the open set defined by $a - r'$ if $r > r'$. Hence, $Sp(A)$ is a compact regular space.

Given a point of $Sp(A)$ that is a model of this theory, we can define $r < \phi(a)$ by $a - r$ and $\phi(a) < s$ by $s - a$. We then have

- $\vdash r < \phi(a), \phi(a) < s$ if $r < s$
- $\vdash \bigvee_s \; \phi(a) < s$
- $\vdash \bigvee_r \; r < \phi(a)$
- $r < \phi(a) \vdash \bigvee_{r'>r} \; r' < \phi(a)$
- $\phi(a) < s \vdash \bigvee_{s'<s} \; \phi(a) < s'$

and hence we have $\phi(a)$ as a real number.

We then check that ϕ is a ring morphism. Notice first that we have $\phi(r) = r$ for any rational r. Indeed since $\vdash s$ if $s > 0$ in Q we have $\phi(r) < s$ if $r < s$ and $s < \phi(r)$ if $s < r$ in Q. That ϕ is an additive morphism follows from $a, b \vdash a + b$. In order to check that ϕ preserves multiplication, one can remark that $4ab = (a + b)^2 - (a - b)^2$ and proves $\phi(a^2) = \phi(a)^2$. This follows from the equivalence, valid if $0 \leq r$ of $r^2 < a^2$ and $r < a \vee a < -r$ and of $a^2 < r^2$ and $a < r \wedge -a < r$. □

This space is called the *spectrum* of R and written $Sp(A)$. We show next that we can consider the elements of R as continuous function on $Sp(A)$, in such a way that the seminorm is preserved.

It is possible to show that the points of $Sp(A)$ can be characterised as the *closed* points of $Sp_{fin}(A)$ [BCR].

3 Gelfand Duality

We shall need the following result, stated in [CC00], which holds for any commutative ring R with a preorder (but not necessarily divisible or archimedian). It is however *essential* for the proof that all square are positive.

Theorem 3.1 *The theory $Sp_{fin}(A)$*

1. $a, -a \vdash$
2. $a + b \vdash a, b$
3. $a \vdash$ *if* $a \leq 0$
4. $\vdash 1$
5. $a, b \vdash ab$
6. $ab \vdash a, -b$

can be characterised as follow

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$$

iff we have a relation $m + p = 0$ where m belongs to the multiplicative monoid generated by $a_1, \ldots, a_n$ and p belongs to the P-cone generated by $a_1, \ldots, a_n, -b_1, \ldots, -b_m$.

Proof. We show that such a relation is an entailment relation [CC00, Sco74]. Only the transitivity is not direct. We show that if M is a multiplicative monoid, if C a P-cone such that $M \subseteq C$ and $x \in R$ is such that we have some relations

$$m_1 + u_1 + (-x)v_1 = 0 \qquad\qquad m_2 x^k + u_2 + x v_2 = 0$$

with $m_1, m_2 \in M$ and $u_1, v_1, u_2, v_2 \in C$ then there is a relation $m+u = 0$ with $m \in M,\ u \in C$. For this let $M' = M + C$ be the set of all elements of the form $m + u,\ m \in M,\ u \in C$. Since $M \subseteq C$ the set M' is closed by multiplication. We can rewrite the first relation as $m_1' = xv_1$ with $m_1' = m_1 + u_1 \in M'$. The second relation implies then $m_2(xv_1)^k + u_2 v_1^k + x v_2 v_1^k = 0$ and hence is of the form $m_2' + xv = 0$ for some $m_2' \in M'$ and $v \in C$. It follows that $m_1' m_2' + x^2 v_1 v = 0$ which is of the form $m' = 0$ for some $m' = m_1' m_2' + x^2 v_1 v \in M'$ as desired.

Conversely let $a_1, \ldots, a_n, b_1, \ldots, b_m \in R$ be given and let M be the multiplicative monoid generated by $a_1, \ldots, a_n$, and C be the P-cone generated by $a_1, \ldots, a_n, -b_1, \ldots, -b_m$. If we have a relation $m + p = 0$ where $m \in M,\ p \in C$ then we can derive

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$$

from the axioms above. This follows from the following two observations.

First, we have the following derived rules

$$a, x + ay \vdash x, y \qquad\qquad x + (-b)y \vdash x, y, b$$

from which we can derive

$$a_1, \ldots, a_n, -p \vdash b_1, \ldots, b_m$$

whenever $p \in C$.

Second, we have

$$a_1, \ldots, a_n \vdash m$$

whenever $m \in M$. $\square$

The next lemma holds for archimedian divisible rings. Notice however that we use only that $r \in P$ if $r \geq 0$ rational, and not that P contains all squares.

Lemma 3.2 *If $1 \leq ab$ and $0 \leq a$ then there exists $r > 0$ such that $r \leq b$.*

Proof. See [Krivine] Théorème 12. In order to be self-contained, and to show that the argument is elementary, we give a sketch of the argument. Since the ring is archimedian, we can find H such that $0 \leq a \leq H$ and K such that $-K \leq b \leq K$. If $a_1 = a/H$ we have then $0 \leq a_1 \leq 1$ and $1/H \leq a_1 b \leq a_1 K$ and hence $1/KH \leq a_1 \leq 1$. If we take $\alpha = 1 - a_1$ we have $0 \leq \alpha \leq 1 - 1/KH$ and $1/H \leq (1-\alpha)b$ It follows from this that we have $1/H \leq (1-\alpha^n)b$ and hence $1/H - \alpha^n K \leq b$ for all n. Since $0 \leq \alpha \leq 1 - 1/KH$ we have $K\alpha^n \leq 1/2H$ for n big enough and hence $1/2H \leq b$. $\square$

Theorem 3.3 *If $\vdash a$ then there exists $r > 0$ such that $r \leq a$ in R.*

Proof. By the proof the theorem 2.1, if we have $\vdash a$ in $Sp(R)$ then we have $\vdash_{fin} \bigvee_{r>0} a - r$ and hence, since $Sp_{fin}(R)$ is finitary, $\vdash_{fin} a - s$ for some $s > 0$.

The P-cone generated by $s - a$ is $P + P(s-a)$. It follows from theorem 3.1 that $\vdash_{fin} a - s$ iff there exists $b, c \geq 0$ such that $1 + b + c(s-a) = 0$. By lemma 3.2 this implies $t \leq a - s$ for some $t > 0$. Hence the result. $\square$

Corollary 3.4 *If $\vdash a \in N(r)$ that is $\vdash r - a$ and $\vdash r + a$ then we have $a \in N(r)$ in A.*

We can consider points of $Sp(A)$ as maps $\phi : A \to R$. If we define $\hat{a} : Sp(A) \to R$ by $\hat{a}(\phi) = \phi(a)$ (usual Gelfand duality) the theorem shows that the uniform norm of $\hat{a}$ is exactly the norm of a in A. That is we have $|\hat{a}(\phi)| < r$ for all $\phi \in Sp(A)$ iff $a \in N(r)$. Notice that it is clear that the elements of A separate the points of $Sp(A)$, so that classically, by Stone-Weirstrass theorem, the elements $\hat{a}$ form a dense subalgebra of $C(Sp(A))$.

4 Theorem of Kadison and Dubois

Another direct corollary is a point-free version of a special case of the theorem of Kadison-Dubois [BS].

Corollary 4.1 *If we have $0 \leq \phi(a)$ for the generic point ϕ of $Sp(A)$ then $1 + an \in P$ for all $n \in N$.*

Proof. For any $n > 0$ we have $\phi(1/n + a) \geq 1/n > 0$ and hence by our theorem we have $a + 1/n \in P$. $\square$

The theorem of Kadison-Dubois is more general in that we don't assume that P contains all square. The goal of this section is to provide a proof of this, giving essentially a localic version of the argument in [BS].

Let R be a divisible ring with a subset P satisfying $r \in P$ if $r \geq 0$ rational, and $P \cdot P \subseteq P$, $P + P \subseteq P$. We can interpret [BS] as follows.

Theorem 4.2 *The following geometrical propositional theory, with atoms $a, b, \ldots$ all elements of R*

1. $a, -a \vdash$
2. $a + b \vdash a, b$
3. $ab \vdash a$ *if* $b \in P$
4. $\vdash 1$
5. $a \vdash \bigvee_{r>0} \quad a - r$

describes a compact regular space: the points of the spaces are the models of this propositional theory, and each $a \in A$ defines the basic open of models satisfying the proposition a. A point of this space defines exactly one ring morphism $\phi : A \to R$ such that $a \geq 0$ implies $\phi(a) \geq 0$.

Theorem 4.3 *We have $a \vdash$ iff $r - a \in P$ for all $r > 0$.*

Proof. The main tool is that of a unitary P-module in R: a subset M satisfying $r \in M$ for $r \geq 0$, and $PM \subseteq M,\ M + M \subseteq M$. We say that $X \subseteq R$ is *inconsistent*, and we write $In(X)$ iff -1 belongs to the unitary P-module generated by X. Let M be the set of finite subsets of R, written $X, Y, \ldots$ We define the relation $X \perp Y$ on M to mean $In(X \cup Y)$. We write X, Y for $X \cup Y$ and X, a for $X \cup \{a\}$. We let $\perp \subseteq M$ be the set of $X \in M$ such that $In(X)$. If $A \subseteq M$ we let $A^{\perp}$ be the set of $X \in M$ such that $X \perp Y$ for all $Y \in A$. Finally we define a *fact* to be a subset A of M such that $A = A^{\perp\perp}$. It is standard that the set of facts form a complete Boolean algebra for the operation

$$0 = \perp \qquad 1 = M \qquad A_1 \wedge A_2 = A_1 \cap A_2 \qquad \bigvee_i A_i = (\bigcup A_i)^{\perp\perp}$$

We are going to give an interpretation of the theory $Sp(R)$ where each formula is interpreted by a fact. The semantics of a is

$$[a] = \{X \in M \mid In(M, -a)\}$$

.

We have to check $[a] \cap [-a] = \perp$. This means that if $In(X, a)$ and $In(X, -a)$ then $In(X)$ and it follows from the proof of lemma 4 in [BS].

The proof of $[a + b] \subseteq ([a] \cup [b])^{\perp\perp}$ follows from the fact that $[a + b] \subseteq [a, b]$.

It is direct that $[ap] \subseteq [a]$ if $p \in P$.

Finally, if we have $In(X, -a)$ then we have $s - ap = -1$ for some s in the unitary M module generated by X and some $p \in P$. Since R is archimedian we can find $r > 0$ such that $s - (a - r)p \leq s < 0$ for some rational $s < 0$ and this implies $In(X, -a + r)$. Hence $[a] \subseteq \cup_{r>0}[a - r]$ and so the last axiom is verified.

In particular if $a \vdash$ we have $[a] = \perp$ that is

$$In(X, -a) \to In(X)$$

Hence we have $In(a - r)$ for all $r > 0$ since clearly $In(a - r, -a)$. We can then find $p, q \in P$ such that $p(r - a) = 1 + q$. We can then use lemma 3.2 noticing that the proof uses only $r \in P$ if $r \geq 0$ rational and not that P contains all squares. $\square$

As shown by Berr and Wörmann [Wörmann], the theorems of Schmüdgen and Polya follow from this result by short, but tricky, algebraic proofs.

A consequence of this theorem is also the following result, which may be not so obvious to prove directly.

Corollary 4.4 *If $a + r \in P$ for all $r > 0$ and $b + r \in P$ for all $r > 0$ then $ab + r \in P$ for all $r > 0$.*

5 Space of Measures

We consider A to be an algebraic presentation of the compact space $Sp(A)$. We can now define a *measure* on $Sp(A)$ to be a linear form $I : A \rightarrow R$ such that $I(1) = 1$ and $0 \leq I(a)$ if $0 \leq a$.

Theorem 5.1 *The space $M(A)$ of measure on $Sp(A)$ is described by the following theory.*

- $a, -a \vdash$
- $a + b \vdash a, b$
- $a \vdash$ *if* $a \leq 0$
- $\vdash 1$
- $a \vdash \bigvee_{r>0} a - r$

It is a compact regular space. A point of this space defines exactly one measure on $Sp(A)$.

We see that points of $Sp(A)$ appear as particular measure (Dirac measure at this point). We have a canonical formal continuous map $M(A) \times A \rightarrow R$ which represents the integral of a function for a given measure.

6 Examples

A first class of examples is provided by any Boolean algebra B. By Stone duality this corresponds to a compact totally disconnected space X [Johnstone] and it is possible to describe directly from B the ring A of Q-valued continuous function $X \rightarrow Q$. We let A to be the Q-algebra generated by symbols $v(b),\ b \in B$ with the relations

$$v(b_1) + v(b_2) = v(b_1 b_2) + v(b_1 \vee b_2) \qquad v(1) = 1 \qquad v(0) = 0$$

The ring A is clearly divisible.

We can define $0 \leq a$ to mean that we can write $a = \Sigma r_i v(b_i)$ with $0 \leq r_i$. Notice that any element $a \in A$ can be written $\Sigma r_i v(b_i)$ with $b_i b_j = 0$ if $i \neq j$. It follows from this remark that we have $0 \leq a^2$ for all a : indeed we have $a^2 = \Sigma r_i^2 v(b_i)$. It is clear also that we have $0 \leq v(b) \leq 1$ for all $b \in B$ and hence that A is archimedian.

Theorem 6.1 *The space $Sp(A)$ is the dual space X of B.*

The construction of this ring A is implicit in [Tarski], and is useful for analysing measures on B.

We give next the example of the space S^1. We take for A the Q-algebra generated by x, y and the relation $x^2 + y^2 = 1$. Using Tarski's completness result for the theory of real closed fields, we can define $0 \leq P(x, y)$ by the fact that the implication

$$x^2 + y^2 = 1 \rightarrow 0 \leq P(x, y)$$

is provable in this theory. (Using elimination of quantifiers for this theory, we see that the order is even decidable in this case.) We get then a divisible archimedian ring A, described in a syntactical way.

Theorem 6.2 *$Sp(A)$ is exactly the space S^1 with its usual topology.*

Conclusion

The duality we get is between compact regular spaces and divisible archimedian preordered rings. It generalises the usual Stone duality between Boolean rings and compact totally disconnected spaces. It should be stressed however that in this constructive setting, the duality is less canonical. In the classical setting or in topos theory, we can consider the Cauchy completion of the ring for its seminormed structure, and we have a categorical correspondance between continuous map and ring map. However, if we do this, we don't represent a space anymore by an object that has a syntactical representation (like it was the case for the space S^1 above for instance). If we want a syntactical description of the space, we need to work with a dense subalgebra of the ring of all continuous functions and there does not seem to be any canonical way to choose such a subring.

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