About Stone's general theory of Spectra

Thierry Coquand

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Two fundamental papers

We present an analysis of the following papers

M.H. Stone "A General Theory of Spectra I" 1940 Proc. N.A.S.

M.H. Stone "A General Theory of Spectra II" 1940 Proc. N.A.S.

Algebraization of spectral theory

"Treatment of any system of real, simultaneously observable quantities as envisaged in the quantum theory"

Analogy with physics

Noticed already in Borel

Note IV Leçons sur la théorie des fonctions

Atomistic approach versus purely phenomenological approach (thermodynamics)

In the notes, Stone gives 2 possible algebraic description of compact Hausdorff spaces in term of observable quantities, not in term of points

Similar goals in formal topology

“Algebraization” of functional analysis

- *formal* description of compact Hausdorff spaces X
- constructive proof of Stone-Weierstrass theorem
- constructive treatment of Gelfad duality (in the real case)
- formal description of the space of probability measure on X
- formal description of Baire subsets of X

Algebraic structures

In both cases, system R of elements $a, b, c, \ldots$

R commutative group

A subset P of "positive" elements: R is an ordered group

A special element 1, so that R is divisible: for each $n > 0$ the equation $nx = 1$ has a solution and R is archimedian: for any $a \in R$ there exists k such that $a \leq k.1$

No "infinitesimal": if $na \leq 0$ for all n then $a \leq 0$

Algebraic structures (continued)

In the *first* note

R is also a commutative ordered ring, such that $a^2 \in P$ for all a

In the *second* note

R is also a l-group: we have a binary l.u.b. operation $a \vee b$

Completeness??

In both cases, we can think of R as a vector space over R and we have a seminormed structure

Define $a \in N(q)$ by $-q' \leq a \leq q'$ for some $q' < q$.

We can complete R so that we have Cauchy completeness

The elements of the completion cannot be "syntactical" objects anymore

In both cases, R will be instead a dense subset of $C(X)$ where X is a compact Hausdorff space (Gelfand duality in the real case)

Spaces in formal topology

"Logical" (opposed to "algebraic") description of a space

We have a set S of "token" or basic observable statements about ideal objects. For instance

$$x < 2,\ 0 < x$$

The finite observations are finite conjunctions of elements of S

A *theory* T over these statements is a set of pairs

$$p_1, \dots, p_n \vdash q_1, \dots, q_m$$

Points as models

A *point* is a model of T, i.e. a subset $\alpha \subseteq S$ such that of $p_1, \ldots, p_n \in \alpha$ and

$$p_1, \ldots, p_n \vdash q_1, \ldots, q_m$$

is in T then at least one q_j is in α

We get then the *spectral* spaces: the space $Sp(T)$ is the set of all models

Basic open subset: $O(p_1, \ldots, p_n)$ set of models satisfying all p_i

General open subset: $O(U) = \bigcup_{X \in U} O(X)$ if U is a set of finite subsets of S

Elimination of points

General form of a *completness* result

We define the relation $X \lhd_\omega U$ directly in term of T. It is defined inductively and holds iff

- there exists $Z \in U$ such that $Z \subseteq X$ or
- there exists $X_1 \vdash Y_1 \in T$ with $X_1 \subseteq X$ and $X, y \lhd_\omega U$ for all $y \in Y_1$

This is similar to the *hyperresolution rule* used in automatic deduction

Theorem: $X \lhd_\omega U$ iff

$$(\forall \alpha \in Sp(T)) \quad \alpha \in O(X) \to \alpha \in O(U)$$

This shows that the logical description of the space is equivalent to its usual description in term of points

Spectral space I

In the case of an ordered ring R we consider the theory T

1. $D(a), D(-a) \vdash$
2. $D(a+b) \vdash D(a), D(b)$
3. $D(a) \vdash$ if $a \leq 0$
4. $\vdash D(1)$
5. $D(a), D(b) \vdash D(ab)$
6. $D(ab) \vdash D(a), D(-b)$

The *models* of this theory define exactly a *total ordering* on R extending the given ordering

Spectral space I (continued)

Theorem: We have

$$D(a_1), \ldots, D(a_n) \vdash_T D(b_1), \ldots, D(b_m)$$

iff we have a relation $m + p = 0$ where m belongs to the multiplicative monoid generated by $a_1, \ldots, a_n$ and p belongs to the P-cone generated by $a_1, \ldots, a_n, -b_1, \ldots, -b_m$.

For instance

$$D(x - y), D(x^2) \vdash D(x^3 - y^3)$$

because

$$0 = (x - y)x^2 + (x - y)(2x^2 + (x + 2y)^2) - 4(x^3 - y^3)$$

$Sp(T)$ is called the *real spectrum* of R

"Phenomenological" description of $Sp(T)$

Example

Proposition: (Krivine) If $1 \leq ab$ and $0 \leq a$ then there exists $r > 0$ such that $r \leq b$

Theorem: If $\vdash_T a$ then $r \leq a$ in R for some $r > 0$

This will give us the *main lemma* in establishing Gelfand's duality

The proof of the theorem is constructive, and similar to arguments used in proof theory (cut-elimination)

Spectral Space II

Similar result, but simpler description of the spectrum, with l-groups

Inspired by F. Riesz "Sur la décomposition des opérations fonctionelles linéaires" 1928

The theory is

1. $D(x), D(-x) \vdash$
2. $D(x) \vdash$ if $x \leq 0$
3. $D(x+y) \vdash D(x), D(y)$
4. $D(x \vee y) \vdash D(x), D(y)$
5. $\vdash D(1)$

Think of $D(x)$ as the proposition $x > 0$

The models are the total order on R extending the given order and preserving binary l.u.b.

Formal consequences

Define, as usual, $x^+ = 0 \vee x$

Theorem: We have

$$D(a_1), \ldots, D(a_n) \vdash_T D(b_1), \ldots, D(b_m)$$

iff there exists N such that

$$1 \wedge a_1^+ \wedge \ldots \wedge a_n^+ \leq N(b_1^+ \vee \ldots b_m^+)$$

Corollary: If $\vdash_T a$ then $r \leq a$ in R for some $r > 0$

Representation of compact spaces

In both cases, to get a description of a compact *Hausdorff* space we have to add the axiom

$$D(a) = \bigcup_{r>0} D(a-r)$$

The *points* are now models α such that

$$a \in \alpha \to (\exists r > 0) a - r \in \alpha$$

The associated formal space is now compact Hausdorff (in the sense of Johnstone's Stone Spaces)

Representation of compact spaces (continued)

Define $a_1, \ldots, a_n \lhd U$ iff for all $r_1, \ldots, r_n > 0$

$$a_1 - r_1, \ldots, a_n - r_n \lhd_\omega U$$

Theorem: We have $a_1, \ldots, a_n \lhd U$ iff

$$(\forall \alpha \in X)) \quad \alpha \in O(X) \rightarrow \alpha \in O(U)$$

Gelfand duality

In the first case X can be also seen as the set of ordered ring morphisms $\phi : R \to \mathbb{R}$

In the second case X can be also seen as the set of l-group morphisms $\phi : R \to \mathbb{R}$

In both case an element a of R defines a real valued function

$$\hat{a} : X \to \mathbb{R} \qquad \phi \longmapsto \phi(a)$$

In both cases we have $\vdash a$ iff $r \leq a$ for some $r > 0$

This shows that the map $R \to C(X)$ is one-to-one and preserves the seminormed structure

Baire subsets of X

We can, in type theory, define the σ-completion R_1 of R. Intuitively we add in a free way l.u.b. of bounded sequences.

If we start from an l-group, we can define a multiplication in R_1

If we start from an ordering ring, we can define a l-group structure in R_1

The elements of R_1 can be identified to the Baire functions on X

Measures

In both cases we can consider the theory

- $D(a), D(-a) \vdash$
- $D(a+b) \vdash D(a), D(b)$
- $D(a) \vdash$ if $a \leq 0$
- $\vdash 1$

Theorem:

$$D(a_1), \dots, D(a_n) \vdash_M D(b_1), \dots, D(b_m)$$

iff we have a relation $r + \Sigma r_i a_i \leq \Sigma s_j b_j$ with r, r_i, s_j rationals all ≥ 0 and $r + \Sigma r_i > 0$.

With the further continuity axiom, the space we obtain is characterised as the space of measures

Generalisation

For commutative ordered rings

point of the space = extremal measures

In the non commutative space, we can still consider extremal measures (pure states). Segal proved that there are enough pure states to separate elements of R. Can we do this constructively??

Conclusion

Real Gelfand duality allows relatively direct constructive proofs

Constructive version of separation theorems: enough points to separate the elements of R

Direct and constructive equivalence of two ways of describing a compact Hausdorff space in term of observable quantities

Next?? One possibility is to look at differential algebra