A COMPLETENESS PROOF FOR GEOMETRICAL LOGIC

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ABSTRACT. Given a geometrical theory, we give a site model, defined as a forcing relation, which is complete for this theory. This model is what is called also the *generic* model of a geometric theory [4]. What is interesting is that this model can be defined without references to logic and the forcing conditions are simply finite sets of atomic formulae, contrary to the model construction in [4, 17]. This model is inspired by [8].

1. GEOMETRICAL THEORY AND GEOMETRICAL LOGIC

A *geometric* or *dynamical* theory is a set of geometric formulae. A *geometric formula* is a first-order formula, without parameters, of the form

$$\phi_0 \to (\exists \overrightarrow{v_1})\phi_1 \vee \cdots \vee (\exists \overrightarrow{v_k})\phi_k$$

where the formulae ϕ_i are finite conjunctions of atomic formulae. There may be free variables present in the formulae ϕ_i and they are, as usual, implicitly universally quantified. We don't assume any range restriction, so there may be free variables appearing in the conclusion not appearing in the hypothesis ϕ_0. A special case is when $k = 0$ in which case the formula becomes $\phi_0 \to \perp$ and expresses the negation of ϕ_0. Another special case is when $k = 1$ and $\overrightarrow{v_1}$ is empty, in which case the formula is of the form $\phi_0 \to \phi_1$ and can be seen as a conjunction of Horn clauses. Finally the conjunction ϕ_0 itself may be empty.

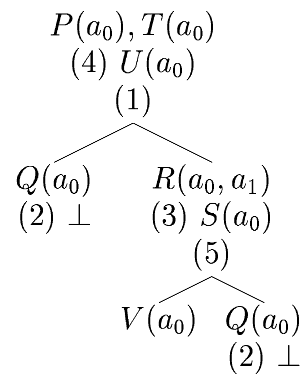
We let $\mathbf{V}$ be the set of all variables $x, y, z, \ldots$ and $\mathbf{P}$ be the set of all parameters $a_0, a_1, \ldots$ Atomic formulae are of the form $R(u_0, \ldots, u_{n-1})$, where $u_0, \ldots, u_{n-1}$ are terms built from variables, parameters and function symbols and R a predicate symbol of arity n. A *sentence* is a closed first-order formula (not necessarily geometric). A *fact* is an atomic sentence, i.e. a closed atomic formula.

In the following we fix a geometric theory T. We now describe the notion of *dynamical proof* with respect to this theory [8]. We look at the formulae of the theory T as a collection of *rules*. The purpose of a dynamical proof is to establish the correctness of a fact with reference to some given set of facts X and the dynamical rules belonging to T starting from a given set of facts. A dynamical proof shows when a given fact F is a consequence of the given set of facts X. Formally, a dynamical proof is a rooted tree. At the root of the tree is the set of facts X we start with. Each node consists of a set of facts, representing a state of information. The sets increase monotonically along the way from the root to the leaves. The successors of a node are determined by the dynamical rules that add new information to the set of already available atomic formulas. The different immediate successors of a node correspond to case distinctions. Every leaf of a dynamical proof contains either a contradiction or the fact under investigation F. If all leaves contain a contradiction then the given set of atomic formulas is contradictory.

Here is an example of a dynamical proof. The geometrical theory is

(1) $P(x) \wedge U(x) \rightarrow Q(x) \vee \exists y.R(x,y)$
(2) $P(x) \wedge Q(x) \rightarrow \perp$
(3) $P(x) \wedge R(x,y) \rightarrow S(x)$
(4) $P(x) \wedge T(x) \rightarrow U(x)$
(5) $U(x) \wedge S(x) \rightarrow V(x) \vee Q(x)$

and the following tree is a derivation of $V(a_0)$ from $P(a_0)$ and $T(a_0)$



The main goal of this note is to show that this method of proof is complete w.r.t. intuitionistic derivation. The method of proof is interesting since it involves a model construction. We are going to build a model of T, which is intuitively a "generic" model. It is defined by a forcing relation, $X \Vdash \phi$, where X is a state of information, i.e. a finite set of facts and where ϕ is now an *arbitrary* first-order sentence. The definition will be such that, if ϕ is a fact, $X \Vdash \phi$ means exactly that there is a dynamical proof of ϕ from X.

2. A Complete Site Model

2.1. **The forcing relation.** The *conditions* will be pairs $X = (I; L)$ where I is a finite set of parameters and L is a finite set of facts, with only parameters in I. We let $D(X)$ be the set of parameters I, and $T(X)$ the set of closed terms built from the parameters in I and $C(X) = L$ the set of facts in X. We write $X \subseteq Y$ iff $D(X) \subseteq D(Y)$ and $C(X) \subseteq C(Y)$.

We define inductively the relation $X \lhd U$ which expresses that a finite set of conditions U *covers* a condition X. The intuition behind this definition is the following: think of X as the initial facts in a dynamical proof, and let $X_0, \ldots, X_{n-1}$ be the set of all branches of this tree, identifying a branch with the finite set of facts appearing in it. Then we have that $X_0, \ldots, X_{n-1}$ cover X. The precise definition is that $X \lhd \{X\}$ and that if $X = (I; L)$ and we have a closed instance of an axiom of T

$$\phi_0 \rightarrow (\exists \overrightarrow{v_1})\phi_1 \vee \cdots \vee (\exists \overrightarrow{v_k})\phi_k$$

with all parameters in I such that all conjuncts of ϕ_0 are in L and for all i

$$(I, \overrightarrow{m_i}; L, \phi_i(\overrightarrow{v_i} = \overrightarrow{m_i})) \lhd U_i$$

where $\overrightarrow{m_i}$ are new parameters not in I, then we have $X \lhd \cup U_i$. It may be that $k = 0$ in which case we have $X \lhd \emptyset$. Notice that if $X \lhd U$ and $Y \in U$ then $X \subseteq Y$.

The conditions should be thought of as finite presentations of a "potential" model of the theory T. A condition $(I; L)$ specifies indeed a finite set of generators I and a finite set of atomic relations L. Given a condition X, a covering $X \lhd U$ can be thought of as a possible finite exploration of a model satisfying X. The branching reflects the fact that there may be non-canonical choices in building the model from the finite information X.

If ϕ is a fact, to say that there is a dynamical proof of ϕ from X means exactly that there exists a covering $X \lhd X_1, \dots, X_n$ with $\phi \in C(X_i)$ for all i.

We define a *map* $f{:}(I; L) \to (J; M)$ between two conditions to be a one-to-one map $f{:}I \to J$ (renaming) such that $\phi f \in M$ if $\phi \in L$. If ϕ is a formula with only parameters in I, we write ϕf the formula obtained by replacing in ϕ the parameter a by the parameter $f(a)$.

If ϕ is a first-order sentence with only parameters in $D(X)$, we define $X \Vdash \phi$ by induction on ϕ.

If ϕ is atomic, $X \Vdash \phi$ if $X \lhd X_0, \dots, X_{n-1}$ and $\phi \in C(X_i)$ for all $i < n$
If ϕ is $\phi_1 \to \phi_2$ we have $X \Vdash \phi$ if for any $f{:}X \to Y$ we have $Y \Vdash \phi_2 f$ whenever $Y \Vdash \phi_1 f$
If ϕ is $\phi_1 \wedge \phi_2$ we have $X \Vdash \phi$ if $X \Vdash \phi_1$ and $X \Vdash \phi_2$
If ϕ is $\phi_1 \vee \phi_2$ we have $X \Vdash \phi$ if $X \lhd U$ and for all $Y \in U$ we have $Y \Vdash \phi_1$ or $Y \Vdash \phi_2$
If ϕ is $(\forall x)\psi$ we have $X \Vdash \phi$ if for any $f{:}X \to Y$ and $a \in T(Y)$ we have $Y \Vdash \psi f(x = a)$
If ϕ is $(\exists x)\psi$ we have $X \Vdash \phi$ if we have $X \lhd X_0, \dots, X_{n-1}$ and $a_i \in T(X_i)$ such that $X_i \Vdash \psi(x = a_i)$ for all $i < n$
If ϕ is $\bot$ we have $X \Vdash \phi$ if $X \lhd \emptyset$

The clause for $X \Vdash (\exists x)\phi$ reflects the fact that we may have to reason by cases to build a witness for an existential statement.

2.2. **Correctness.** The main result is the following.

Theorem 2.1. *If $\vdash_T \phi$ then for any $\rho{:}\mathbf{P} \to T(X)$ we have $X \Vdash \phi\rho$. More generally, if $\phi_1, \dots, \phi_n \vdash_T \phi$ and we have $X \Vdash \phi_1\rho, \dots, X \Vdash \phi_n\rho$ then $X \Vdash \phi\rho$.*

Lemma 2.2. *If $X \lhd U$ and $f{:}X \to Y$ then there exists V such that $Y \lhd V$ and for all $Y' \in V$ there exists $X' \in U$ with $g{:}L \to M$ which extends f.*

Proof. We prove this by induction on the construction of $X \lhd U$.

If $U = \{X\}$ we can take $V = \{Y\}$.

If $X = (I; L)$ and there exists a closed instance of an axiom of T

$$\phi_0 \to (\exists \overrightarrow{v_1})\phi_1 \vee \dots \vee (\exists \overrightarrow{v_k})\phi_k$$

such that all conjuncts of ϕ_0 are in L and for all i

$$X_i = (I, \overrightarrow{m_i}; L, \phi_i(\overrightarrow{v_i} = \overrightarrow{m_i})) \lhd U_i$$

we extend the renaming f by choosing $\overrightarrow{b_i}$ not in $D(Y)$ and by taking $g_i(\overrightarrow{m_i}) = \overrightarrow{b_i}$. We then define

$$Y_i = (D(Y), \overrightarrow{b_i}; L, \phi_i(\rho, \overrightarrow{v_i} = \overrightarrow{b_i}))$$

so that $g_i{:}X_i \to Y_i$. By induction hypothesis, we can find V_i such that $Y_i \lhd V_i$ and for all $Y' \in V_i$ there exists $X' \in U_i$ with $h : X' \to Y'$ which extends g_i. We then take $V = \cup V_i$. □

Lemma 2.3. *If $X \lhd X_0, \ldots, X_{n-1}$ and $X_i \lhd V_i$ for all $i < n$ then $X \lhd \cup_{i<n} V_i$.*

Proof. This is direct by induction on the proof of $X \lhd X_0, \ldots, X_{n-1}$. □

Lemma 2.4. *If $X \Vdash \phi$ and $f{:}X \to Y$ then $Y \Vdash \phi f$.*

Proof. We prove this by induction on ϕ.

If ϕ is atomic we have U such that $X \lhd U$ and $\phi \in C(Z)$ for all $Z \in U$. By lemma 2.2 we can find V such that $Y \lhd V$ and for all $T \in V$ there is $Z \in U$ and $g : Z \to T$ which extends f. Since $\phi \in C(Z)$ we have $\phi g \in C(T)$ and since g extends f and ρ takes its values in $T(X)$ we have $\phi g = \phi f$. It follows that we have $\phi f \in C(T)$ for all $T \in V$ and hence $Y \Vdash \phi f$.

If ϕ is $\phi_1 \to \phi_2$ and $g{:}Y \to Z$ we have $Z \Vdash \phi_2 f g$ whenever $Z \Vdash \phi_1 f g$ and hence $Y \Vdash \phi f$. (Notice that we don't use any induction hypothesis in this case.)

If ϕ is $\phi_1 \wedge \phi_2$ we have, by induction hypothesis, $Y \Vdash \phi_1 f$ and $Y \Vdash \phi_2 f$ and so $Y \Vdash \phi f$.

If ϕ is $\phi_1 \vee \phi_2$ we have U such that $X \lhd U$ and for all $Z \in U$ we have $Z \Vdash \phi_1$ or $Z \Vdash \phi_2$. By lemma 2.2 we can find V such that $Y \lhd V$ and for all $T \in V$ there is $Z \in U$ and $g : Z \to T$ which extends f. By induction hypothesis, we have then $T \Vdash \phi_1 g$ or $T \Vdash \phi_2 g$ since g extends f and ρ takes its values in $T(X)$, and hence $\phi_i g = \phi_i f$. Thus for all $T \in V$ we have $T \Vdash \phi_1 f$ or $T \Vdash \phi_2 f$ and hence $Y \Vdash \phi f$.

If ϕ is $(\forall x)\psi$ we have for any $g{:}Y \to Z$ and $a \in T(Z)$ that $Z \Vdash \psi f g(x = a)$. Hence, $Y \Vdash \phi f$. (Notice that we don't use any induction hypothesis in this case.)

If ϕ is $(\exists x)\psi$ we have U such that $X \lhd U$ and for all $Z \in U$ we have $Z \Vdash \psi(x = a)$ for some $a \in T(Z)$. By lemma 2.2 we can find V such that $Y \lhd V$ and for all $T \in V$ there is $Z \in U$ and $g : Z \to T$ which extends f. We have $Z \Vdash \psi(x = a)$ for some $a \in T(Z)$ and so, by induction hypothesis, $T \Vdash \psi g(x = g(a))$. Since g extends f and ρ takes its values in $T(X)$ we have $\psi g = \psi f$. Thus we get $T \Vdash \psi f(x = g(a))$. This shows that $Y \Vdash \phi f$.

If ϕ is $\perp$ we have $X \lhd \emptyset$. By lemma 2.2 we have also $Y \lhd \emptyset$ and so $Y \Vdash \phi f$. □

Lemma 2.5. *If $X \lhd X_0, \ldots, X_{n-1}$ and $X_i \Vdash \phi$ for all $i < n$ then $X \Vdash \phi$.*

Proof. We prove this by induction on ϕ.

If ϕ is atomic we have for all $i < n$ a set V_i such that $X_i \lhd V_i$ and $\phi \in C(Y)$ for all $Y \in V_i$. By lemma 2.3 we have $X \lhd \cup_{i<n} V_i$ and hence $X \Vdash \phi$.

If ϕ is $\phi_1 \to \phi_2$ and $f : X \to Y$ is such that $Y \Vdash \phi_1 f$ we have by lemma 2.2 V such that $Y \lhd V$ and for all $Z \in V$ there exists $i < n$ and $g : X_i \to Z$ extending f. By lemma 2.4

we have $Z \Vdash \phi_1 f$. Since g extends f we have $\phi_1 f = \phi_1 g$ and so $Z \Vdash \phi_1 g$. Since $X_i \Vdash \phi$ we get that $Z \Vdash \phi_2 g$ and hence $Z \Vdash \phi_2 f$. Since this holds for all $Z \in V$ we get by induction hypothesis $Y \Vdash \phi_2 f$. Hence $X \Vdash \phi$.

If ϕ is $\phi_1 \wedge \phi_2$ we have by induction hypothesis $X \Vdash \phi_1$ and $X \Vdash \phi_2$ and so $X \Vdash \phi$.

If ϕ is $\phi_1 \vee \phi_2$ we have for all $i < n$ a set V_i such that $X_i \lhd V_i$ and $Y \Vdash \phi_1$ or $Y \Vdash \phi_2$ for all $Y \in V_i$. By lemma 2.3 we have $X \lhd \cup_{i<n} V_i$ and hence $X \Vdash \phi$.

If ϕ is $(\forall x)\psi$ and $f : X \to Y$ and $a \in T(Y)$ we have by lemma 2.2 V such that $Y \lhd V$ and for all $Z \in V$ there exists $i < n$ and $g : X_i \to Z$ extending f. Then $Z \Vdash \psi g(x = a)$. Since g extends f we have $\psi f = \psi g$ and so $Z \Vdash \psi f(x = a)$. Since this holds for all $Z \in V$ we get by induction hypothesis $Y \Vdash \psi f(x = a)$. Hence $X \Vdash \phi$.

If ϕ is $(\exists x)\psi$ we have for all $i < n$ a set V_i such that $X_i \lhd V_i$ and for all $Y \in V_i$ we have $m \in T(Y)$ such that $Y \Vdash \psi(x = m)$. By lemma 2.3 we have $X \lhd \cup_{i<n} V_i$ and hence $X \Vdash \phi$.

If ϕ is $\perp$ we have $X_i \lhd \emptyset$ for all $i < n$ and by lemma 2.3 we get $X \lhd \emptyset$ and so $X \Vdash \phi$. $\square$

We can now prove the main theorem. If Γ is a finite set of sentences and ϕ is a sentence we define inductively $\Gamma \vdash \phi$ by the clauses (this is a convenient formulation of the usual intuitionistic natural deduction for first-order logic)

(1) $\Gamma \vdash \phi$ if $\phi \in \Gamma$
(2) $\Gamma \vdash \phi_1 \to \phi_2$ if $\Gamma, \phi_1 \vdash \phi_2$
(3) $\Gamma \vdash \phi_1 \wedge \phi_2$ if $\Gamma \vdash \phi_1$ and $\Gamma \vdash \phi_2$
(4) $\Gamma \vdash \phi_1 \vee \phi_2$ if $\Gamma \vdash \phi_1$ or $\Gamma \vdash \phi_2$
(5) $\Gamma \vdash (\forall x)\psi$ if $\Gamma \vdash \psi(x = a)$ for some fresh parameter a
(6) $\Gamma \vdash (\exists x)\psi$ if $\Gamma \vdash \psi(x = t)$ for some term t
(7) $\Gamma \vdash \phi_2$ if $\Gamma \vdash \phi_1 \to \phi_2$ and $\Gamma \vdash \phi_1$
(8) $\Gamma \vdash \phi_i$ if $\Gamma \vdash \phi_1 \wedge \phi_2$
(9) $\Gamma \vdash \phi$ if $\Gamma \vdash \phi_1 \vee \phi_2$ and $\Gamma, \phi_i \vdash \phi$
(10) $\Gamma \vdash \phi$ if $\Gamma \vdash (\exists x)\psi$ and $\Gamma, \psi(x = a) \vdash \phi$ for some fresh a
(11) $\Gamma \vdash \phi(x = t)$ if $\Gamma \vdash (\forall x)\phi$
(12) $\Gamma \vdash \phi$ if $\Gamma \vdash \perp$

If $\Gamma = \{\phi_1, \ldots, \phi_n\}$ we let $X \Vdash \Gamma$ mean that $X \Vdash \phi_i$ for all $i = 1, \ldots, n$.

We now prove that if $X \Vdash \Gamma\rho$ then $X \Vdash \phi\rho$ by induction on the proof of $\Gamma \vdash \phi$.

Suppose $\phi \in \Gamma$ then $X \Vdash \Gamma\rho$ implies directly $X \Vdash \phi\rho$.

If ϕ is $\phi_1 \to \phi_2$ and $\Gamma, \phi_1 \vdash \phi_2$. Assume $X \Vdash \Gamma\rho$ and $f{:}X \to Y$ and $Y \Vdash \phi_1\rho f$. By lemma 2.4 we have $Y \Vdash \Gamma\rho f$. Hence $Y \Vdash (\Gamma, \phi_1)\rho f$. Hence by induction $Y \Vdash \phi_2\rho f$. This shows $X \Vdash \phi\rho$.

If ϕ is $\phi_1 \wedge \phi_2$ and $\Gamma \vdash \phi_1$, $\Gamma \vdash \phi_2$ and $X \Vdash \Gamma\rho$, by induction, we have $X \Vdash \phi_1\rho$ and $X \Vdash \phi_2\rho$ and hence $X \Vdash \phi\rho$.

If ϕ is $\phi_1 \vee \phi_2$ and $\Gamma \vdash \phi_1$ or $\Gamma \vdash \phi_2$ and $X \Vdash \Gamma\rho$, by induction, we have $X \Vdash \phi_1\rho$ or $X \Vdash \phi_2\rho$ and hence, since $X \lhd \{X\}$, we have $X \Vdash \phi\rho$.

If ϕ is $(\forall x)\psi$ and $\Gamma \vdash \psi(x = a)$ for some fresh a and $f{:}X \to Y$ and $m \in T(Y)$ then, by lemma 2.4, we have $Y \Vdash \Gamma\rho f$. We define $\nu{:}\mathbf{P} \to T(Y)$ by taking $\nu(u) = f(\rho(u))$ if $u \neq a$ and $\nu(a) = m$. We have by induction $Y \Vdash \psi(x = a)\nu$ which is $Y \Vdash \psi\rho f(x = m)$. This shows $X \Vdash \phi\rho$.

If ϕ is $(\exists x)\psi$ and $\Gamma \vdash \psi(x = t)$ for some term t and $X \Vdash \Gamma\rho$ we have, by induction, $X \Vdash \psi(x = t)\rho$ which is $X \Vdash \psi\rho(x = t\rho)$. Since $X \lhd \{X\}$, we get $X \Vdash \phi\rho$.

If $\Gamma \vdash \psi \to \phi$ and $\Gamma \vdash \psi$ and $X \Vdash \Gamma\rho$ then, by induction, we have $X \Vdash (\psi \to \phi)\rho$ and $X \Vdash \psi\rho$ which implies $X \Vdash \phi\rho$.

If $\Gamma \vdash \phi_1 \wedge \phi_2$ and $X \Vdash \Gamma\rho$ then, by induction, we have $X \Vdash (\phi_1 \wedge \phi_2)\rho$ and hence $X \Vdash \phi_i\rho$ for $i = 1, 2$.

If $\Gamma \vdash \phi_1 \vee \phi_2$ and $\Gamma, \phi_i \vdash \phi$ for $i = 1, 2$ and $X \Vdash \Gamma\rho$ then, by induction we have $X \Vdash (\phi_1 \vee \phi_2)\rho$. Hence we have $X \lhd X_0, \ldots, X_{n-1}$, with $X_i \Vdash \phi_1\rho$ or $X_i \Vdash \phi_2\rho$ for each $i < n$. Since $X \subseteq X_i$ we have by lemma 2.4 that $X_i \Vdash \Gamma\rho$. Also $X_i \Vdash (\Gamma, \phi_1)\rho$ or $X_i \Vdash (\Gamma, \phi_2)\rho$. By induction, this implies $X_i \Vdash \phi\rho$. By lemma 2.5, we get $X \Vdash \phi\rho$.

Suppose $\Gamma \vdash (\exists x)\psi$ and $\Gamma, \psi(x = a) \vdash \phi$ with a fresh and $X \Vdash \Gamma\rho$. By induction we have $X \Vdash ((\exists x)\psi)\rho$. Hence we have $X \lhd X_0, \ldots, X_{n-1}$ and $m_i \in T(X_i)$ with $X_i \Vdash \psi\rho(x = m_i)$. Since $X \subseteq X_i$ we have by lemma 2.4 that $X_i \Vdash \Gamma\rho$. If we define $\nu_i{:}\mathbf{P} \to T(Y)$ by $\nu_i(u) = \rho(u)$ if $u \neq a$ and $\nu_i(a) = m_i$ we have $\Gamma\nu_i = \Gamma\rho$ and so $X_i \Vdash \Gamma\nu_i$ and $X_i \Vdash \psi(x = a)\nu_i$ since $\psi(x = a)\nu_i = \psi\rho(x = m_i)$. Hence by induction $X_i \Vdash \phi\nu_i$. Hence for all i we have $X_i \Vdash \phi\rho$ since $\phi\rho = \phi\nu_i$. It follows that we have $X \Vdash \phi\rho$ by lemma 2.5.

If $\Gamma \vdash (\forall x)\phi$ and $X \Vdash \Gamma\rho$ then, by induction, we have $X \Vdash (\forall x)\phi\rho$. This implies $X \Vdash \phi\rho(x = t\rho)$ which is $X \Vdash \phi(x = t)\rho$.

If $\Gamma \vdash \perp$ and $X \Vdash \Gamma\rho$ then, by induction, $X \Vdash \perp \rho$ and hence $X \lhd \emptyset$. By lemma 2.5 this implies $X \Vdash \phi\rho$.

This concludes the proof of the main theorem.

2.3. Simplification.

Lemma 2.6. *If we have that $Y \Vdash \phi_1$ implies $Y \Vdash \phi_2$ for all $Y \supseteq X$ then $X \Vdash \phi_1 \to \phi_2$. If $Y \Vdash \phi(x = a)$ for all $Y \supseteq X$ and $a \in T(Y)$ then $X \Vdash (\forall x)\phi$.*

Proof. We treat only the case of implication, since the case of universal quantification has a similar justification. Assume that $Y \Vdash \phi_1$ implies $Y \Vdash \phi_2$ for all $Y \supseteq X$ and that $f{:}X \to Y$ is such that $Y \Vdash \phi_1 f$. There exists then $Y_0 \supseteq X$ with a *bijective* map $f_0 : Y_0 \to Y$ extending f. By lemma 2.4, we have $Y_0 \Vdash \phi_1 f f_0^{-1}$. Since f_0 extends f this implies $Y_0 \Vdash \phi_1$. Hence by hypothesis, we have also $Y_0 \Vdash \phi_2$. By lemma 2.4, this implies $Y \Vdash \phi_2 f_0$ and since f_0 extends f we have also $Y \Vdash \phi_2 f$ as desired. □

This shows that in the definition of the forcing relation, for the clauses for implication and universal quantification, we can limit ourselves to renamings that are inclusions. Hence the definition of forcing can be stated without references to renaming. A similar remark is made in [2].

2.4. Propositional case.

The definition of forcing simplifies: the conditions are now finite sets of atomic propositions

(1) $X \Vdash \phi$ if $X \lhd U$ and $\phi \in C(Y)$ for all $Y \in U$
(2) $X \Vdash \phi_1 \to \phi_2$ if for any $Y \supseteq X$ we have $Y \Vdash \phi_2$ whenever $Y \Vdash \phi_1$
(3) $X \Vdash \phi_1 \wedge \phi_2$ if $X \Vdash \phi_1$ and $X \Vdash \phi_2$
(4) $X \Vdash \phi_1 \vee \phi_2$ if $X \lhd U$ and for all $Y \in U$ we have $Y \Vdash \phi_1$ or $Y \Vdash \phi_2$

(5) $X \Vdash \bot$ if $X \lhd \emptyset$

and our definition of $X \Vdash F$ becomes similar to the one of hyper-resolution [18]. The method of trees in this case can be traced back to Lewis Carroll [1].

2.5. **Related work.** Our definition of the syntactical site is similar to the one presented in [17, 4]. However one main difference is that our notion of morphism in this site is simply *renaming* and hence does not refer to the theory T, as in these references. This is important for instance if we want to use our model to show the *consistency* of the theory T. In [16] there is another construction, attributed to Coste, closer to our definition, which is also given in [8]. There morphisms are algebra morphisms, and the objects are finitely presented structures of a suitable subtheory of the theory T. It is not emphasized however there that this gives a purely syntactical, and constructive, completeness proof of the notion of dynamical proof for geometrical formulae[1]. A related construction is presented in [2], attributed to Buchholz, which applies to any theory (not necessarily geometrical). In the references [5] and [7], we present a completeness proof for topological models using a *generalised* inductive definition. It is remarkable that the present completeness proof uses only ordinary inductive definitions.

Our completeness theorem can be compared to theorem 1.1 of [8], which is a cut-elimination theorem. Both results can be seen as algorithms to transform a usual proof into a dynamical proof. It would be interesting to compare these two algorithms on simple examples.

The notion of dynamical proof is quite close to the tableau method [21]. Since it is possible to write any first-order formula in a geometrical way, essentially by naming each subformula and its negation, our completeness result actually shows also the completeness of the tableau method. In [5], we present an example showing the possible interest of the notion of dynamical proof for automatic deduction. Similar ideas, with an implementation in Prolog, appeared already in [15].

3. Examples

3.1. **Infinite model.** The following theory

$\neg(x < x)$
$x < y \wedge y < z \rightarrow x < z$
$(\exists y)[x < y]$

is consistent but has no finite model. In this case, finite presentations define finite posets and we can build directly a forcing model by taking finite posets as conditions. A direct extension of a poset X is obtained by choosing $x \in X$ and adding a new element y to X with the only constraint that $y > x$. We write $X \lhd Y$ if we get Y from X by successive direct extensions. The forcing relation becomes

$X \Vdash \phi$ if ϕ holds in X
$X \Vdash \phi_1 \rightarrow \phi_2$ if for any map $f{:}X \rightarrow Y$ we have $Y \Vdash \phi_2 f$ whenever $Y \Vdash \phi_1 f$

[1]For instance, in [8] a similar construction is presented as a non-constructive model construction.

$X \Vdash \phi_1 \wedge \phi_2$ if $X \Vdash \phi_1$ and $X \Vdash \phi_2$
$X \Vdash \phi_1 \vee \phi_2$ if $X \lhd Y$ and we have $Y \Vdash \phi_1$ or $Y \Vdash \phi_2$
$X \Vdash (\forall x)\phi$ if for any map $f{:}X \to Y$ and $a \in Y$ we have $Y \Vdash \phi f(x = a)$
$X \Vdash (\exists x)\phi$ if $X \lhd Y$ and we have $Y \Vdash \phi(x = a)$ for some $a \in Y$

Furthermore, though the theory has no finite models, the consistency is established by considering only finitely presented, and hence in this case finite, structures. This seems connected to similar remarks in [19].

3.2. **Theory of fields.** The theory of fields has terms built from $1, 0, +, -, \times$ and only one predicate symbols $Z(t)$, which stands for $t = 0$. We can then write $t_1 = t_2$ for $Z(t_1 - t_2)$ and can consider the terms modulo the usual equations for rings. We have the three axioms for rings

$Z(0)$
$Z(a) \wedge Z(b) \to Z(a + b)$
$Z(a) \to Z\ (ab)$

In order to get the theory of fields we add the axioms $\neg Z(1)$ and

$$Z(x) \vee (\exists y)Z(xy - 1)$$

The conditions can be thought of as finite presentations of rings. We can then simplify the site model by taking as conditions finitely presented rings and as morphisms finitely presented extensions (adding finitely many new parameters and new equations). Starting from a ring A with an element $a \in A$ the basic covering corresponding to the axiom of field is then obtained by taking the two extensions $A \to A/\langle a \rangle$ and $A \to A[x]/\langle ax - 1 \rangle$.

Another possible geometric axiom that we can add to the theory of ring is

$$(\exists y)(xy = 1) \vee (\exists y)((1 - x)y = 1)$$

which expresses that the ring is a *local* ring. In this case the basic coverings are obtained by the two extensions $A \to A[x]/\langle ax - 1 \rangle$ and $A \to A[x]/\langle (1 - a)x - 1 \rangle$.

Here is a remark, due to Kock [14], which shows an interesting consequence of the main theorem 2.1. The following non geometrical formula is forced in this theory

$$\Vdash \neg(\wedge x_i = 0) \to \vee_i (\exists y)(x_i y = 1) \qquad (*)$$

Indeed, we have $A \Vdash \neg(\wedge x_i = 0)$ iff $1 \in \langle x_0, \dots, x_{n-1} \rangle$ in A.[2] It is also clear that if $1 \in \langle x_0, \dots, x_{n-1} \rangle$ in A then we have $\vee_i (\exists y)(x_i y = 1)$ if A is a local ring. It follows then from the main theorem that if a geometrical formula can be proved with $(*)$ then it can be proved without.

[2]For this it is enough to consider the finitely presented extension $A \to A/\langle x_0, \dots, x_{n-1} \rangle$ which corresponds to adding the facts $Z(x_0), \dots, Z(x_{n-1})$. This shows that to have $A \Vdash \neg(\wedge x_i = 0)$ implies that the ring $A/\langle x_0, \dots, x_{n-1} \rangle$ is trivial.

3.3. **Consistency versus Quantifier Elimination.** As seen in the two examples above, one interest of the method is to allow the construction of models, and hence to analyse the consistency of a theory. This may be interesting even if the theory admits quantifier elimination, because the consistency proof may be simpler than the proof of quantifier elimination. We believe that Herbrand had something similar in mind when he alluded to a proof of quantifier elimination for proving the consistency of the theory of real closed fields and then added that his model construction provides a simpler consistency argument [10].

We shall treat the example of the theory of algebraically closed fields. The argument is reminiscent of the one used by Skolem [20] in his analysis of the theory of the projective plane. In both cases, the crucial step is to show that the introduction of "auxiliary elements" allowed by existential axiom does not prove new facts about the old elements.

Lemma 3.1. *If $a, b \in A$ then b is nilpotent in $A[x]/<ax-1>$ iff ab is nilpotent in A.*

Proof. If b^m is 0 in $A[x]/<ax-1>$ it is 0 in $A[1/a]$. This implies that for some n we have $a^n b^m = 0$ in A and hence ab is nilpotent. □

Corollary 3.2. *If $a, b \in A$ and b is nilpotent in $A[x]/<ax-1>$ and in $A/<a>$ then b is nilpotent in A.*

Lemma 3.3. *If p is a monic non-constant polynomial in $A[x]$ and $a \in A$ then a is nilpotent in A if, and only if, it is nilpotent in $A[x]/<p>$.*

Proof. Since p is monic and non-constant, an equality $a^n = pq$ for $q \in A[x]$ implies $q = 0$ and hence $a^n = 0$ in A. □

The geometric theory of algebraically closed fields is obtained from the theory of rings by adding to the axiom of fields the axiom schema, for $n \geq 1$

$$(\exists x) Z(x^n + a_{n-1}x^{n-1} + \cdots + a_0)$$

The forcing conditions are arbitrary finitely presented rings, and we add as a basic covering the extension $A \to A[x]/<p>$ for each monic and non-constant $p \in A[x]$.

Theorem 3.4. *If $a \in A$ then $A \Vdash Z(a)$ iff a is nilpotent.*

Proof. Direct from corollary 3.2 and lemma 3.3. □

It follows from our main theorem that, for any ring A and any formula intuitionistically provable from the (geometric) theory of algebraically closed fields and the positive diagram of A we have $A \Vdash \phi$.

In particular, suppose that $Z(a)$ is derivable from the geometric theory of algebraically closed fields and the positive diagram of A. Then we have $A \Vdash Z(a)$ and hence a is nilpotent. Thus, if $Z(1)$ is derivable A should be a trivial ring. This shows the consistency of the theory of algebraically closed fields. Furthermore this consistency proof can be interpreted as building effectively a *non-standard* model of the theory.

In the reference [12] is sketched an argument for the consistency of this theory which proves also quantifier elimination. A complete argument is presented in [8]. In the context of our paper, the result of quantifier elimination can be interpreted as follows: the consistency of a branch in a dynamical proof in the theory of algebraically closed fields is *decidable.*

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