Internal version of the uniform Kan filling condition

Introduction

We present a notion of fibration of cubical sets. This is formulated in term of the notion of partial element, which has a natural semantics in a presheaf model. We define a partial element to be connected if it can be extended to a total element. (The justification of this terminology is that this would generalize in the present framework the notion of two points being connected by a path.) To be fibrant can then be defined internally as the fact that if a partial path is connected at 0 then it also is connected at 1.

Cubical sets

Base category

Let \( C \) the following category. The objects are finite sets \( I, J, \ldots \). A morphism \( \text{Hom}(J, I) \) is a map \( I \to dM(J) \) where \( dM(J) \) is the free de Morgan algebra on \( J \). We write \( f : J \to I \) for \( f \in \text{Hom}(J, I) \).

We write \( 1_I : I \to I \) the identity map of \( I \). If \( f : J \to I \) and \( g : K \to J \) we write \( fg : K \to I \) their composition.

A presheaf \( X \) on \( C \) can be described as a family of sets \( X(I) \) together with restriction maps \( X(I) \to X(J) \), \( u \mapsto uf \) for \( f : J \to I \), satisfying \( u1_I = u \) and \( (uf)g = u(fg) \). (This notation for the restriction map is motivated by the canonical isomorphism between \( X(I) \) and \( I \to X \), where \( I \) is the presheaf represented by \( I \).)

The presheaf \( I \) is defined by \( I(J) = dM(J) \). We can think of an element of \( dM(I) \) as a lattice formula \( \psi \) on atoms \( i, 1 - i \) for \( i \) in \( I \). If \( f : J \to I \), and \( \psi \) is in \( dM(I) \), then the restriction operation \( \psi f \) can be thought as a substitution: we replace the atom \( i \) by \( f(i) \) in the formula \( \psi \).

A sieve on \( I \) is a collection \( L \) of maps of codomain \( I \) such that \( fg \in L \) whenever \( f : J \to I \) is in \( L \) and \( g : K \to J \). If \( L \) is a sieve on \( I \) then we let \( Lf \) be the sieve on \( J \) of all maps \( g : K \to J \) such that \( fg \) is in \( L \). We define in this way a presheaf \( \Omega \), taking \( \Omega(I) \) be the set of sieves on \( I \). Each element \( \psi \) in \( dM(I) \) determines the sieve of \( f : J \to I \) such that \( \psi f = 1 \). This defines a natural transformation \( I \to \Omega \).

In each \( dM(I) \) there is a greatest element \( < 1 \), the disjunction of all \( i \) and \( 1 - i \) for \( i \) in \( I \). The sieve associated to this element is the boundary of \( I \).

A cubical set is a presheaf on \( C \).

Partial elements and connectedness

\( \Omega \) is internally the set of truth-values. To each element \( p \) in \( \Omega \) we can associate a subobject \([p]\) of the constant cubical set \( 1 \). A partial element of a cubical set \( X \) can be defined as a pair \( p, u \) where \( p \) is in \( \Omega \) and \( u \) is a map \([p] \to X \). The element \( p \) is called the extent of the partial element \( p, u \). (Alternatively, a partial element of \( X \) can be defined as a subsingleton of \( X \).

We think of \( I \) as a formal representation of the real interval \([0, 1]\). It has a structure of a de Morgan algebra. The map \( I \to \Omega \) can be described internally as the map \( i \mapsto i = 1 \), which associates to an element \( i \) the truth-value \( i = 1 \). Using the disjunction property of free de Morgan algebra (which results from the conjunctive normal form representation of formulae), we see that we have

\[
(i \land j = 1) = (i = 1) \land (j = 1) \quad \quad (i \lor j = 1) = (i = 1) \lor (j = 1)
\]

and the map \( I \to \Omega \), \( i \mapsto i = 1 \) is an injective lattice map. We identify \( \psi \) with the truth-value \( \psi = 1 \).
Lemma 0.1 If we have $\psi : I \rightarrow I$ we can define $\forall i.\psi(i)$ in $I$ such that

$$(1 = \forall i.\psi(i)) = \forall i : I (1 = \psi(i))$$

Proof. This corresponds to a map $dM(I, i) \rightarrow dM(I)$ natural in $I$. We let $(\forall i.\psi(i))(I)$ be the disjunction of the conjunctions not mentioning $i$ in the disjunctive normal form of $\psi(i)$ in $dM(I, i)$. \qed

If $\psi$ is an element in $\mathbb{I}$ and $u$ is a partial element of $X$ of extent $\psi$, we write $X[\psi \mapsto u]$ the subset of $X$ of element in $X$ that extends $u$. An element of this set is a witness that $u$ is connected.

Fibrations

A family of sets $A\rho$ for $\rho$ in $\Gamma$ is a fibration iff we have an operation which takes as argument a path $\gamma$ in $\Gamma^1$, an element $\psi$ in $I$, a partial section $u(i)$ of $A\gamma(i)$ of extent $\psi$, an element in $A\gamma(0)[\psi \mapsto u(0)]$, and produces an element in $A\gamma(1)[\psi \mapsto u(1)]$. This operation is thus an element of

$$(\gamma : \Gamma^1) \ (\psi : \mathbb{I}) \ (u : ((i : \mathbb{I}) \rightarrow A\gamma(i))[\psi]) \rightarrow A\gamma(0)[\psi \mapsto u(0)] \rightarrow A\gamma(1)[\psi \mapsto u(1)]$$

Lemma 0.2 If we have a composition operation

$$\text{comp} : (\gamma : \Gamma^1) \ (\psi : \mathbb{I}) \ (u : ((i : \mathbb{I}) \rightarrow A\gamma(i))[\psi]) \rightarrow A\gamma(0)[\psi \mapsto u(0)] \rightarrow A\gamma(1)[\psi \mapsto u(1)]$$

then we have a filling operation: given $\gamma$ in $\Gamma^1$, $\psi$ in $\mathbb{I}$, $u$ in $((i : \mathbb{I}) \rightarrow A\gamma(i))[\psi]$ and $a_0$ in $A\gamma(0)[\psi \mapsto u(0)]$, we can find a section in

$$(i : \mathbb{I}) \rightarrow A\gamma(i)[\psi \mapsto u(i), (1 - i) \mapsto a_0]$$

Proof. We define

$$\text{fill} \ \gamma \ \psi \ u \ a_0 \ i = \text{comp} \ \gamma \ (\psi \vee (1 - i)) \ v \ a_0$$

where $v$ is the partial element in $((j : \mathbb{I}) \rightarrow A\gamma(j))[\psi \vee (1 - i)]$ which is equal to $\lambda j. u(i \land j)$ on $[\psi]$ and $\lambda j. a_0$ on $[i = 0]$. This is well-defined since $u(i \land j) = u(0) = a_0$ on $[\psi] \cap [i = 0]$. \qed

Taking as a special case 0 for $\psi$, we see that if $\Gamma \vdash A$ is a fibration then we have the path lifting property: we have an operation taking as argument $\gamma$ in $\Gamma^2$ and $a_0$ in $A\gamma(0)$ and producing a section $a(i) : A\gamma(i)$ such that $a(0) = a_0$.

We say that a cubical set $A$ is fibrant if it defines a fibration over the constant cubical set 1. Explicitly, it means that we have an operation taking as argument an element $\psi$ in $\mathbb{I}$, a partial path $u$ in $A^{\mathbb{I}}$ of extent $\psi$ and producing a map $A[\psi \mapsto u(0)] \rightarrow A[\psi \mapsto u(1)]$. The previous Lemma shows that we then have another operation producing an element in

$$(a_0 : A[\psi \mapsto u(0)]) \rightarrow (i : \mathbb{I}) \rightarrow A[\psi \mapsto u(i), (1 - i) \mapsto a_0]$$

Model of type theory

Proposition 0.3 If we have fibrations $\Gamma \vdash A$ and $\Gamma, x : A \vdash B$ then $\Gamma \vdash (x : A) \rightarrow B$ is a fibration.

Proof. Let us write $C = (x : A) \rightarrow B$. Given $\gamma$ in $\Gamma^2$ and $\psi$ in $\mathbb{I}$ and $\mu$ in $((i : \mathbb{I}) \rightarrow C\gamma(i))[\psi]$ and $\lambda_0$ in $C\gamma(0)[\psi \mapsto \mu(0)]$, we define $\lambda_1 : C\gamma(1)[\psi \mapsto \mu(1)]$ by taking

$$\lambda_1 \ a_1 = \text{comp} \ (\lambda i. (\gamma(i), a(i))) \ (\psi \mapsto \mu(i) \ a(i)) \ (\lambda_0 \ a(0))$$

where $a(i) : A\gamma(i)$ satisfying $a(1) = a_1$ is defined using the path lifting property of $\Gamma \vdash A$. \qed

We have a similar definition for the dependent sum $\Gamma \vdash (x : A, B)$. If $\Gamma \vdash A$ and we have two sections $\Gamma \vdash u : A$ and $\Gamma \vdash v : A$ then we define $\Gamma \vdash \text{Id} \ A \ u \ v$ by taking $(\text{Id} \ A \ u \ v)\rho$ to be the subset of path $p$ in $(A\rho)^2$ such that $p(0) = u\rho$ and $p(1) = v\rho$. 

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Proof. We suppose given γ in Γ^ι and p_0 in (ld A u v)_\gamma(0) in ι and a partial section q(i) in (ld A u v)_\gamma(i) of extent ψ such that q(0) = p_0 on ψ. This means that we have q(i, j) in A_\gamma(i) and q(i, 0) = \gamma_\psi(i) and q(i, 1) = \gamma_\psi(i). We define then p_1 in (ld A u v)_\gamma(1) by p_1(j) = comp_\gamma_\psi(j \gamma_\psi j \gamma_\psi(1 - j)) r p_0(j) where r(i) is a partial section in A_\gamma(i) of extent ψ ∨ j ∨ (1 - j) defined as r(i) = q(i, j) on [ψ] and r(i) = \gamma_\psi(i) on [1 - j] and r(i) = \gamma_\psi(i) on [j].

\[\square\]

Isomorphisms

If T and A are two cubical sets, an isomorphism T → A consists in two maps f : T → A and g : A → T and two sections s : (x : T) → ld T (g (f x)) x and t : (x : A) → ld A (f (g x)) x. So we have a map s : T × I → T such that s(x, 0) = g (f x) and s(x, 1) = x and a map t : A × I → A such that t(x, 0) = f (g x) and t(x, 1) = x.

Lemma 0.5 If T and A are fibrant, and we have an isomorphism (f, g, s, t) : T → A then we have an operation taking as argument ψ in I and a partial element \tilde{t} in T of extent ψ and a in A[ψ ↦ f t] and producing an element in (x : T, ld A a (f x))[ψ ↦ (t, 1_a)].

Glueing operation

Lemma 0.6 We assume given a section Γ ⊢ σ : T → A where Γ ⊢ A, Γ ⊢ T are two fibrations. Given γ in Γ^ι and a partial section t(i) ∈ Tγ(i) of extent ψ and t_0 in Tγ(0)[ψ ↦ t(0)], we can consider a_0 = σγ(0) t_0 in A_γ(0) and the partial section a(i) = σγ(i) t(i) of extent ψ. There is a path connecting a_1 = comp_A γ ψ a a_0 to σγ(1) t_1 where t_1 = comp_T γ ψ t_0 in Tγ(1). This path is furthermore constant on the extent ψ.

Proof. By filling in T, we find an extension of the partial section t to a total section \tilde{t} such that \tilde{t}(1) = t_1. By filling in A, we find an extension of the partial section a to a total section \tilde{a} such that \tilde{a}(1) = a_1. Given i we define the partial section u of extent ϕ = ψ ∨ (1 - i) ∨ i by taking u(j) = σγ(j) t(j) on ψ and u(j) = \tilde{a}(j) on i = 0 and u(j) = σγ(j) t(j) on i = 1. The path joining a_1 to σγ(1) t_1 is then \lambda i.\text{comp}_A ϕ u a_0.

If Γ ⊢ ψ : I, i.e. we have ψ : Γ → ι, we define Γ, ψ to be the subset of elements ρ in Γ such that ψ(ρ) = 1.

The rules for the glueing operation are

\[\frac{\Gamma ⊢ A \quad \Gamma, ϕ ⊢ T \quad Γ, ϕ ⊢ σ : \text{Is}(T, A)}{\Gamma ⊢ \text{glue}(A, [ϕ ↦ (T, σ)])} \quad \frac{\Gamma ⊢ \text{glue}(A, [1 ↦ (T, σ)]) = T}{\Gamma ⊢ \text{glue}(A, [ϕ ↦ (T, σ)]) = T}\]

We write B = glue(A, [ϕ ↦ (T, σ)]) and we explain the composition operation for B.

If ρ in Γ any element of Bρ can be written uniquely on the form glue(a, [ϕρ ↦ t]) with a in Aρ, t a partial element of Tρ of extent ϕρ such that ϕρ t = a.

We assume given γ in Γ^ι, and element b_0 = (a_0, [ϕγ(0) ↦ t_0]) and a partial section w(i) = (u(i), [ϕγ(i) ↦ w(i)]) of extent ψ. We want to define b_1 = (a_1, [ϕγ(1) ↦ t_1]) in B_γ(1)[ϕ ↦ v(1)].

We first consider a_0 in A_γ(0) and u(i) in A_γ(i) of extent ψ, and such that a_0 = u(0). Since Γ ⊢ A is a fibration, we get a_1' in A_γ(1), such that a_1' = u(1) on ψ.

Using Lemma 0.1 we define δ = ϕ i [ϕγ(i) in ι. We have δ ≤ ϕγ(1). On the extent ψ ∨ δ we can consider t_0 in Tγ(0) and the partial section w(i) in Tγ(i). Since Γ, ϕ ⊢ T is a fibration we define t_1' in Tγ(1) of extent δ and such that t_1' = w(1) on δ ∨ ψ. Using Lemma 0.6, we have a path between a_1' and σγ(1) t_1' of extent δ. Since A_γ(1) is fibrant we can then find a_1'' in A_γ(1) such that a_1'' = a_1' on ψ and a_1'' = σγ(1) t_1' on δ.
Using the fact that $\sigma\gamma(1)$ is an isomorphism and Lemma 0.5, we can extend $t'_1$ in $T\gamma(1)$ to an element $t_1$ of extent $\varphi\gamma(1)$ such that $t_1 = w(1)$ on $\psi$. Using that $A\gamma(1)$ is fibrant, we find $a_1$ in $A\gamma(1)$ such that $\sigma\gamma(1) t_1 = a_1$ on $\varphi\gamma(1)$ and and $a_1 = a''_1 = a'_1$ on $\psi$. The element $b_1 = \text{glue}(a_1,[\varphi\gamma(1) \mapsto t_1])$ is in $B\gamma(1)[\psi \mapsto v(1)]$ and satisfies $b_1 = t'_1$ on the extent $\delta$. 