Topology and Sequent Calculus

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Sequent Calculus

Gentzen (1932) analysed a version of a sequent calculus designed by Hertz (1922)

These sequents are of the form

$$x_1, \ldots, x_n \vdash x$$

where $x_1, \ldots, x_n, x$ are "events"

Intuitive meaning:

if $x_1, \ldots, x_n$ hold simultaneously, so does x

The axioms are

- $X \vdash x$ if $x \in X$
- $X' \vdash x$ if $X \subseteq X'$ et $X \vdash x$
- $X \vdash y$ if $X \vdash x$ and $X, x \vdash y$

Information System

Dana Scott uses this sequent calculus for a presentation of domains (1980) : a domain is a suitable *theory*

The “events” represent finite informations that we have on possibily infinite objects

These objects are the *models* of the theory representing the domain

Example of reals (may be partial)

$$x \in (r, s) \vdash x \in (r', s')$$

if $r' \leq r$ and $s \leq s'$

An example: Functions

$N \to \{0, 1\}$ is represented by the theory

$$(n, 0), (n, 1) \vdash$$

The *models* are exactly the *partial functions* from N to $\{0, 1\}$

A model is a subset $\alpha \subseteq S$ such that if $X \subseteq \alpha$ and $X \vdash Y$ then Y meets α. Here $S = N \times \{0, 1\}$

Entailment Relations

An important generalisation, due to Gentzen, is *multi-conclusion* sequent calculus

Dana Scott has given an abstract version of this notion, which appears as a quite natural generalisation of Hertz' calculus

$X \vdash Y$ where X, Y are both *finite sets*

Intuitively

$$x_1, \ldots, x_n \vdash y_1, \ldots, y_m$$

means: if all x_i hold simultaneously then at least one of the events y_j holds

Remarks

It is an abstract notion: $\vdash$ is not necessarily definied inductively, and S is not a priori a set of syntactical formulae

In particular, there may be no logical connectives $\wedge$, $\vee$, $\neg$ defined on the set S

We can *characterise* the existence of a connector $\wedge$: it should be a function $\wedge : S \times S \rightarrow S$ such that

$$X, a_1, a_2 \vdash Y \quad ssi \quad X, a_1 \wedge a_2 \vdash Y$$

Remarks

The "elimination" property

$$\frac{X, a \vdash Y \qquad X \vdash Y, a}{X \vdash Y}$$

is a property of $\vdash$, which may be shown in a non inductive way

A theory described the set of its models. It is a concrete and logical representation of some topological space

Example: function space

If we consider the theory

$$(n, 0), (n, 1) \vdash$$

$$\vdash (n, 0), (n, 1)$$

the models are now exactly the *total* functions from N to $\{0, 1\}$

Ordered Group

Let A be a commutative group. For each $x \in A$ we have the atoms $x > 0$. The theory is

- $x > 0, -x > 0 \vdash$
- $x + y > 0 \vdash x > 0, y > 0$

Theorem: $X \vdash Y$ iff we have $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$ such that $n > 0$ and $\Sigma x_i = \Sigma y_j$

A model of $\vdash$ defines a *total* ordering of A

Example: Zariski Spectrum

Let A be a commutative ring with unit. The theory is

- $\vdash 0 \in P$
- $x \in P, y \in P \vdash x + y \in P$
- $x \in P \vdash xy \in P$
- $1 \in P \vdash$
- $xy \in P \vdash x \in P, y \in P$

$\vee$ can be defined by

$$x \in P \vee y \in P = xy \in P$$

Example: Zariski Spectrum

A *model* is exactly a *prime ideal* of A

This theory is a concrete presentation of the Zariski spectrum of A

Theorem: We have

$$x_1 \in P, \ldots, x_n \in P \vdash y_1 \in P, \ldots, y_m \in P$$

iff the monoid generated by $y_1 \ldots y_m$ meets the ideal $(x_1, \ldots, x_n)$

This is a formal version of Hilbert's Nullstellensatz: if $f_1, \ldots, f_n$ have no common zero then 1 is in $(f_1, \ldots, f_n)$

This shows in an elementary way the consistency of the theory $Sp(A)$, without having to exhibit a model

Conservativity = surjectivity

Given two entailment relations on S and S' and $f : S \to S'$ such that

$$X \vdash Y \to f(X) \vdash f(Y)$$

then any model $\beta \subseteq S'$ induces a model $f^{-1}(\beta) \subseteq S$

f is an *interpretation* between two theories

Theoreme: $\beta \longmapsto f^{-1}(\beta)$ is surjective iff

$$f(X) \vdash f(Y) \to X \vdash Y$$

i.e. iff the interpretation f is *conservative*

Simple consequence of the completness theorem

Extension theorems

Any map $f : A \to A'$ on induces $f^* : Sp(A') \to Sp(A)$

If f is integral, for any prime ideal of A there is a prime ideal of A' lying over it

f^* is *surjective* in this case

This can be proved in a purely algebraic and elementary way

Adding negation

We add new atoms $a \notin P$ for $a \in A$

- $\vdash a \in P, a \notin P$
- $a \in P, a \notin P \vdash$

$\vdash$ can be characterised as

$$a_i \in P, b_j \notin P \vdash c_k \in P, d_l \notin P$$

iff

$$a_i \in P, d_j \in P \vdash c_k \in P, b_j \in P$$

iff the monoid generated by c_k, b_j meets the ideal (a_i, d_j)

This defines the *constructible topology* of A

Tarski-Chevalley

A. Joyal (1975) has established a connection between Chevalley's theorem on constructible topology and Skolem-Tarski's quantifier elimination procedure

We say that $f : A \to A'$ is *finitely presented* iff it is a composition of maps $A \to A[x]$ or $A \to A/(a)$

Theorem: The interpretation $B(A) \to B(A')$ admits an existential quantifiers $\exists_f : B(A') \to B(A)$ (i.e. a map such that $x \leq f^*(y)$ iff $\exists_f(x) \leq y$)

In topological term the map f^* is *open* (Chevalley's theorem)

Real Spectrum

- $\vdash 1 > 0$
- $x > 0, -x > 0 \vdash$
- $x > 0, y > 0 \vdash xy > 0$
- $x + y > 0 \vdash x > 0, y > 0$
- $xy > 0 \vdash x > 0, -x > 0$
- $xy > 0 \vdash x > 0, -y > 0$

We have $X \vdash Y$ iff $m + p = 0$ for some m in the monoid generated by X X and p is in the positive cone generated by X and $\{-y \mid y \in Y\}$

Real Spectrum

For instance

$$x - y > 0, x^2 > 0 \vdash x^3 - y^3 > 0$$

because

$$0 = (x-y)x^2 + (x-y)(2x^2 + (x+2y)^2) - 4(x^3 - y^3)$$

(Also

$$x - y > 0 \vdash x^3 - y^3 > 0$$

exercice: compute the corresponding identity...)

The measure problem

Let M be a commutative monoid. Suppose $a_0 \in M$ is such that, for all $a \in M$ there exists n and x such that $a + x = n.a_0$

The atoms are $a < b$ for $a, b \in M$ and the theory is

- $a < b, b < a \vdash$
- $a + b < c + d \vdash a < c, b < d$
- $a < b \vdash a + x < b + x$

The measure problem

Theorem: $X \vdash Y$ iff there is $x_i < y_i \in X,\ i \leq n$, and $z_j < t_j \in Y$ with $n > 0$ and x such that

$$\Sigma y_i + \Sigma z_j + x = \Sigma x_i + \Sigma t_j + x$$

A model defines $f : M \to R$ additive such that $f(a_0) = 1$. Hence such f "exists" iff there is no n and x such that

$$(n + 1).a_0 + x = n.a_0$$

(This is a result of Tarski 1929)

Hahn-Banach theorem

C. Mulvey (1980) formulates Hahn-Banach as a conservativity theorem the atoms are $a \in (r, s)$ with $a \in E$ and $r, s \in Q$

- $\vdash 0 \in (r, s)$ if $r < 0 < s$
- $0 \in (r, s) \vdash \perp$ if $0 \leq r$ or $s \leq 0$
- $a \in (r, s) \vdash -a \in (-s, -r)$
- $a \in (r, s) \vdash ta \in (tr, ts)$ if $t > 0$
- $a \in (r, s), a' \in (r', s') \vdash a + a' \in (r + r', s + s')$
- $a \in (r, s) \vdash a \in (r, s'), a \in (r', s)$ if $r' < s'$
- $\vdash a \in (-1, 1)$ if $a \in N(1)$
- $a \in (r', s') \vdash a \in (r, s)$ if $r \leq r' < s' \leq s$

Solution with entailment relations

If $E_1 \subseteq E_2$ then $T(E_2)$ conservative over $T(E_1)$

We decompose $a \in (r, s)$ in

$$a < s \quad \wedge \quad r < a$$

and $a < s$ is defined as $-s < -a$

- $r < x, x < r \vdash$
- $r + s < x + y \vdash r < x, s < y$
- $\vdash x < 1$ if $x \in N(1)$

We can characterise simply $\vdash$ and get a direct proof of conservativity

Valuations

If K is a field, a *valuation ring* A is a subring of K such that if x invertible $x \in A$ or $x^{-1} \in A$

The atoms are $x \in A$ for x *invertible*

- $\vdash x \in A, y \in A$ si $xy = x + y$
- $\vdash x \in A, x^{-1} \in A$
- $x \in A \vdash -x \in A$
- $x \in A, y \in A \vdash xy \in A$

Valuations

Theorem: We have

$$x_1 \in A, \ldots, x_n \in A \vdash y_1 \in A, \ldots, y_m \in A$$

iff there exist $q_1, \ldots, q_m$ such that

$$y_1^{-1} q_1 + \cdots + y_m^{-1} q_m = 1$$

In particular $x_1 \in A, \ldots, x_n \in A \vdash y \in A$ iff y is *integral* over A

Sylvester elimination theorem is used

Valuations

If $a_i,\ i \leq n$ and $b_j,\ j \leq m$ indeterminates

$c_k = \Sigma a_i b_j$ then $a_i b_j$ is integral over $c_0, \ldots, c_{n+m}$ (Kronecker's theorem)

Related Works

A. Jung, M.A.M. Moshier and M. Kegelmann. "Multi lingual sequent calculus and coherent spaces." Fundamenta Informaticae, vol 37, 1999, pages 369-412

M. Coste, H. Lombardi, and M.F. Roy. "Dynamical method in algebra: Effective nullstellensätze." Preprint, 1996