Cubical Type Theory

1 Syntax

Γ, Δ	$::=$	$() \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \varphi$	Contexts
r, s	$::=$	$0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s$	The interval, $\mathbb{I}$
φ, ψ	$::=$	$0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$	The face lattice, $\mathbb{F}$
t, u, A, B	$::=$	$x \mid \lambda x : A.\, t \mid t\ u \mid (x : A) \to B$	Π-types
	$\mid$	$(t, u) \mid t.1 \mid t.2 \mid (x : A) \times B$	Σ-types
	$\mid$	$\mathsf{0} \mid \mathsf{s}\, u \mid \mathsf{natrec}\ t\ u \mid \mathsf{N}$	Natural numbers
	$\mid$	$\langle i \rangle\ t \mid t\ r \mid \mathsf{Path}\ A\ t\ u$	Path types
	$\mid$	S	
	$\mid$	$\mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0$	Compositions
	$\mid$	$\mathsf{glue}\ [\varphi \mapsto t]\ u \mid \mathsf{unglue}\ [\varphi \mapsto (T, f)]\ u \mid \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A$	Glueing
	$\mid$	U	Universe
S	$::=$	$[\ \varphi_1\ t_1, \ldots, \varphi_n\ t_n\ ]$	Systems

2 Inference rules

Well-formed contexts, $\Gamma \vdash$

$$\frac{}{() \vdash} \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}\ (x \notin \mathrm{dom}(\Gamma)) \qquad \frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash}\ (i \notin \mathrm{dom}(\Gamma)) \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash}$$

Well-typed elements of $\mathbb{I}$, $\Gamma \vdash r : \mathbb{I}$

$$\frac{}{\Gamma \vdash 0 : \mathbb{I}} \qquad \frac{}{\Gamma \vdash 1 : \mathbb{I}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash i : \mathbb{I}}\ (i : \mathbb{I} \in \Gamma) \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash 1 - r : \mathbb{I}}$$

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I}}{\Gamma \vdash r \wedge s : \mathbb{I}} \qquad \frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I}}{\Gamma \vdash r \vee s : \mathbb{I}}$$

Well-typed elements of $\mathbb{F}$, $\Gamma \vdash \varphi : \mathbb{F}$

$$\frac{}{\Gamma \vdash 0_{\mathbb{F}} : \mathbb{F}} \qquad \frac{}{\Gamma \vdash 1_{\mathbb{F}} : \mathbb{F}} \qquad \frac{\Gamma \vdash i : \mathbb{I}}{\Gamma \vdash (i = 0) : \mathbb{F}} \qquad \frac{\Gamma \vdash i : \mathbb{I}}{\Gamma \vdash (i = 1) : \mathbb{F}}$$

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma \vdash \psi : \mathbb{F}}{\Gamma \vdash \varphi \wedge \psi : \mathbb{F}} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma \vdash \psi : \mathbb{F}}{\Gamma \vdash \varphi \vee \psi : \mathbb{F}}$$

Well-formed types, $\Gamma \vdash A$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \times B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{N}} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{Path}\ A\ t\ u}$$

$$\frac{\Gamma \vdash \vee_i \varphi_i = 1_{\mathbb{F}} : \mathbb{F} \qquad \Gamma, \varphi_i \vdash A_i \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash A_i = A_j}{\Gamma \vdash [\ \varphi_1\ A_1, \ldots, \varphi_n\ A_n\ ]}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \mathsf{Equiv}\ T\ A}{\Gamma \vdash \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{U}} \qquad \frac{\Gamma \vdash A : \mathsf{U}}{\Gamma \vdash A}$$

Well-typed terms, $\Gamma \vdash t : A$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\ t : (x : A) \to B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x : A \in \Gamma)$$

$$\frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(x/u)} \qquad \frac{\Gamma \vdash t : (x : A) \times B}{\Gamma \vdash t.1 : A} \qquad \frac{\Gamma \vdash t : (x : A) \times B}{\Gamma \vdash t.2 : B(x/t.1)}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u) : (x : A) \times B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{0} : \mathsf{N}} \qquad \frac{\Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{s}\ n : \mathsf{N}}$$

$$\frac{\Gamma, x : N \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\ n)}{\Gamma \vdash \mathsf{natrec}\ a\ b : (x : \mathsf{N}) \to P}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle\ t : \mathsf{Path}\ A\ t(i0)\ t(i1)} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\ r : A}$$

$$\frac{\Gamma \vdash \vee_i \varphi_i = 1_{\mathbb{F}} : \mathbb{F} \qquad \Gamma \vdash A \qquad \Gamma, \varphi_i \vdash t_i : A \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash t_i = t_j : A}{\Gamma \vdash [\ \varphi_1\ t_1, \ldots, \varphi_n\ t_n\ ] : A}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

$$\frac{\Gamma \vdash b : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}{\Gamma \vdash \mathsf{unglue}\ b : A[\varphi \mapsto f\ b]} \qquad \frac{\Gamma, \varphi \vdash f : \mathsf{Equiv}\ T\ A \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\ t]}{\Gamma \vdash \mathsf{glue}\ [\varphi \mapsto t]\ a : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

$$\frac{\Gamma \vdash A : \mathsf{U} \qquad \Gamma, x : A \vdash B : \mathsf{U}}{\Gamma \vdash (x : A) \to B : \mathsf{U}} \qquad \frac{\Gamma \vdash A : \mathsf{U} \qquad \Gamma, x : A \vdash B : \mathsf{U}}{\Gamma \vdash (x : A) \times B : \mathsf{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{N} : \mathsf{U}}$$

$$\frac{\Gamma \vdash A : \mathsf{U} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{Path}\ A\ t\ u : \mathsf{U}}$$

$$\frac{\Gamma \vdash A : \mathsf{U} \qquad \Gamma, \varphi \vdash T : \mathsf{U} \qquad \Gamma, \varphi \vdash f : \mathsf{Equiv}\ T\ A}{\Gamma \vdash \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A : \mathsf{U}}$$

Interval equality, $\Gamma \vdash r = s : \mathbb{I}$

$$\frac{r = s\ (\mathsf{mod.}\ \Gamma)}{\Gamma \vdash r = s : \mathbb{I}}$$

Face lattice equality, $\Gamma \vdash \varphi = \psi : \mathbb{F}$

$$\frac{\varphi = \psi\ (\mathsf{mod.}\ \Gamma)}{\Gamma \vdash \varphi = \psi : \mathbb{F}}$$

Type equality, $\Gamma \vdash A = B$ (Congruence and equivalence rules are omitted)

$$\frac{\Gamma \vdash \varphi_i = 1_{\mathbb{F}} : \mathbb{F}}{\Gamma \vdash [\ \varphi_1\ A_1, \ldots, \varphi_n\ A_n\ ] = A_i} \qquad \frac{\Gamma \vdash T \qquad \Gamma \vdash f : \mathsf{Equiv}\ T\ A}{\Gamma \vdash \mathsf{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A = T}$$

Term equality, $\Gamma \vdash a = b : A$ (Congruence and equivalence rules are omitted)

$$\frac{\Gamma \vdash t = u : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t = u : B} \qquad \frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x : A.\ t)\ u = t(x/u) : B(x/u)}$$

$$\frac{\Gamma, x : A \vdash t\ x = u\ x : B}{\Gamma \vdash t = u : (x : A) \to B} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u).1 = t : A} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u).2 = u : B(x/t)}$$

$$\frac{\Gamma \vdash t.1 = u.1 : A \qquad \Gamma \vdash t.2 = u.2 : B(x/t.1)}{\Gamma \vdash t = u : (x : A) \times B}$$

$$\frac{\Gamma, x : \mathsf{N} \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\ n)}{\Gamma \vdash \mathsf{natrec}\ a\ b\ 0 = a : P(x/0)}$$

$$\frac{\Gamma, x : \mathsf{N} \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\, n) \qquad \Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{natrec}\ a\ b\ (\mathsf{s}\ n) = b\ n\ (\mathsf{natrec}\ a\ b\ n) : P(x/\mathsf{s}\, n)}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash (\langle i \rangle\ t)\ r = t(i/r) : A} \qquad \frac{\Gamma, i : \mathbb{I} \vdash t\ i = u\ i : A}{\Gamma \vdash t = u : \mathsf{Path}\ A\ u_0\ u_1} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\ 0 = u_0 : A}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\ 1 = u_1 : A} \qquad \frac{\Gamma \vdash [\ \varphi_1\ t_1, \ldots, \varphi_n\ t_n\ ] : A \qquad \Gamma \vdash \varphi_i = 1_{\mathbb{F}} : \mathbb{F}}{\Gamma \vdash [\ \varphi_1\ t_1, \ldots, \varphi_n\ t_n\ ] = t_i : A}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{glue}\ [1_{\mathbb{F}} \mapsto t]\ a = t : T} \qquad \frac{\Gamma \vdash b : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}{\Gamma \vdash b = \mathsf{glue}\ [\varphi \mapsto b]\ (\mathsf{unglue}\ b) : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

$$\frac{\Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\ t]}{\Gamma \vdash \mathsf{unglue}\ (\mathsf{glue}\ [\varphi \mapsto t]\ a) = a : A}$$

Together with this family of rules for an arbitrary judgment body J:

$$\frac{\Gamma \vdash \vee_i \varphi_i = 1_{\mathbb{F}} : \mathbb{F} \qquad \Gamma, \varphi_1 \vdash \mathsf{J} \qquad \cdots \qquad \Gamma, \varphi_n \vdash \mathsf{J}}{\Gamma \vdash \mathsf{J}}$$

3 Equality judgments for comp

Before defining the equality judgments for compositions we define some useful operations:

Kan filling:

$$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i/i \wedge j)\ [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$$

where j is fresh for Γ.

Transport: Composition for $\varphi = 0_{\mathbb{F}}$ corresponds to transport:

$$\Gamma \vdash \mathsf{transp}^i\ A\ a = \mathsf{comp}^i\ A\ []\ a : A(i1)$$

Contr: Given $\Gamma \vdash p : \mathsf{isContr}\ A$ and $\Gamma, \varphi \vdash u : A$ we define the operation:

$$\Gamma \vdash \mathsf{contr}\ p\ [\varphi \mapsto u] = \mathsf{comp}^i\ A\ [\varphi \mapsto p.2\ u\ i]\ p.1 : A[\varphi \mapsto u]$$

Pres: Given

$$\Gamma, i : \mathbb{I} \vdash f : T \to A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash t : T \qquad \Gamma \vdash t_0 : T(i0)[\varphi \mapsto t(i0)]$$

Let $\Gamma \vdash a_0 = f(i0)\ t_0 : A(i0)$ and $\Gamma, \varphi, i : \mathbb{I} \vdash a = f\ t : A$, together with:

$$\Gamma, i : \mathbb{I} \vdash u = \mathsf{fill}^i\ A\ [\varphi \mapsto a]\ a_0 : A \qquad \Gamma, i : \mathbb{I} \vdash v = \mathsf{fill}^i\ T\ [\varphi \mapsto t]\ t_0 : T$$

we then define

$$\Gamma \vdash \mathsf{pres}^i\ f\ [\varphi \mapsto t]\ t_0 = \langle j \rangle\ \mathsf{comp}^i\ A\ [\varphi \mapsto a, (j = 0) \mapsto u, (j = 1) \mapsto f\ v]\ a_0$$

of type $(\mathsf{Path}\ A(i1)\ c_1\ c_2)[\varphi \mapsto \langle j \rangle\ (f\ t)(i1)]$ where $c_1 = \mathsf{comp}^i\ A\ [\varphi \mapsto f\ t]\ (f(i0)\ t_0)$ and $c_2 = f(i1)\ (\mathsf{comp}^i\ T\ [\varphi \mapsto t]\ t_0)$.

Equiv: Given

$$\Gamma \vdash f : \mathsf{Equiv}\ T\ A \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A \qquad \Gamma, \varphi \vdash p : \mathsf{Path}\ A\ a\ (f\ t)$$

we define

$$\Gamma \vdash \mathsf{equiv}\ f\ [\varphi \mapsto (t,p)]\ a = \mathsf{contr}\ (f.2\ a)\ [\varphi \mapsto (t,p)] : (x:T) \times \mathsf{Path}\ A\ a\ (f\ x)[\varphi \mapsto (t,p)]$$

The equality judgments for $\mathsf{comp}^i\ C\ [\varphi \mapsto u]\ a_0$ are defined by cases on the type C:

- **Product types**, $C = (x : A) \to B$:
  Let

  $$\Gamma, i : \mathbb{I} \vdash w = \mathsf{fill}^i\ A(i/1-i)\ []\ u_1 : A(i/1-i) \quad \text{and} \quad \Gamma, i : \mathbb{I} \vdash v = w(i/1-i) : A.$$

  Using this we define

  $$\Gamma \vdash (\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 = \mathsf{comp}^i\ B(x/v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ v(i0)) : B(i1)[\varphi \mapsto (\mu\ v)(i1)]$$

- **Sum types**, $C = (x : A) \times B$
  Let $\Gamma, i : \mathbb{I} \vdash a = \mathsf{fill}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 : A$ and

  $$\Gamma \vdash c_1 = \mathsf{comp}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 \qquad \Gamma \vdash c_2 = \mathsf{comp}^i\ B(x/a)\ [\varphi \mapsto w.2]\ w_0.2$$

  Using this we define:

  $$\Gamma \vdash \mathsf{comp}^i\ C\ [\varphi \mapsto w]\ w_0 = (c_1, c_2) : C(i1)[\varphi \mapsto w(i1)]$$

- **Natural numbers**, $C = \mathsf{N}$:

  $$\Gamma \vdash \mathsf{comp}^i\ C\ [\varphi \mapsto \mathsf{0}]\ \mathsf{0} = \mathsf{0} : C[\varphi \mapsto \mathsf{0}]$$
  $$\Gamma \vdash \mathsf{comp}^i\ C\ [\varphi \mapsto \mathsf{s}\ n]\ (\mathsf{s}\ n_0) = \mathsf{s}\ (\mathsf{comp}^i\ C\ [\varphi \mapsto n]\ n_0) : C[\varphi \mapsto \mathsf{s}\ n]$$

- **Path types**, $C = \mathsf{Path}\ A\ u\ v$:

  $$\Gamma \vdash \mathsf{comp}^i\ C\ [\varphi \mapsto p]\ p_0 = \langle j \rangle\ \mathsf{comp}^i\ A\ S\ (p_0\ j) : C(i1)[\varphi \mapsto p(i1)]$$

  where the system S is $[\varphi \mapsto p\ j, (j=0) \mapsto u, (j=1) \mapsto v]$.

- **Glue**, $C = \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A$:
  We assume

  $$\Gamma, \psi, i : \mathbb{I} \vdash b : C \qquad \Gamma \vdash b_0 : C(i0)[\psi \mapsto b(i0)]$$

  and define:

  $$\Gamma, \psi, i : \mathbb{I} \vdash a = \mathsf{unglue}\ b : A[\varphi \mapsto f\ b]$$
  $$\Gamma \vdash a_0 = \mathsf{unglue}\ b_0 : A(i0)[\varphi(i0) \mapsto f(i0)\ b_0, \psi \mapsto a(i0)]$$

  The following provides the algorithm for composition $b_1 = \mathsf{comp}^i\ C\ [\psi \mapsto b]\ b_0$ of type $C(i1)[\psi \mapsto b(i1)]$.

  $$\begin{array}{lcll}
  \delta & = & \forall i.\varphi & \Gamma \\
  a'_1 & = & \mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0 & \Gamma \\
  t'_1 & = & \mathsf{comp}^i\ T\ [\psi \mapsto b]\ b_0 & \Gamma, \delta \\
  \omega & = & \mathsf{pres}^i\ f\ [\psi \mapsto b]\ b_0 & \Gamma, \delta \\
  (t_1, \alpha) & = & \mathsf{equiv}\ f(i1)\ [\delta \mapsto (t'_1, \omega), \psi \mapsto (b(i1), \langle j \rangle a'_1)]\ a'_1 & \Gamma, \varphi(i1) \\
  a_1 & = & \mathsf{comp}^j\ A(i1)\ [\varphi(i1) \mapsto \alpha\ j, \psi \mapsto a(i1)]\ a'_1 & \Gamma \\
  b_1 & = & \mathsf{glue}\ [\varphi(i1) \mapsto t_1]\ a_1 & \Gamma
  \end{array}$$

- **Universe**, $C = \mathsf{U}$:

  Given $\Gamma \vdash A$, $\Gamma \vdash B$, and $\Gamma, i : \mathbb{I} \vdash E$, such that $E(i0) = A$ and $E(i1) = B$, we will construct $\mathsf{equiv}^i\ E : \mathsf{Equiv}\ A\ B$. In order to do this we first define

$$\begin{array}{rcl} \Gamma \vdash f & = & \lambda x : A.\ \mathsf{transp}^i\ E\ x : A \to B \\ \Gamma \vdash g & = & \lambda y : B.\ (\mathsf{transp}^i\ E(i/1-i)\ y)(i/1-i) : B \to A \\ \Gamma, i : \mathbb{I} \vdash u & = & \lambda x : A.\ \mathsf{fill}^i\ E\ []\ x : A \to E \\ \Gamma, i : \mathbb{I} \vdash v & = & \lambda y : B.\ (\mathsf{fill}^i\ E(i/1-i)\ []\ y)(i/1-i) : B \to E \end{array}$$

  We will now prove that f is an equivalence. Given $y : B$ we see that $(x : A) \times \mathsf{Path}\ B\ y\ (f\ x)$ is inhabited as it contains the element $(g\ y, \gamma)$ where

$$\gamma = \langle j \rangle\ \mathsf{comp}^i\ E\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ (g\ y)]\ (g\ y)$$

  We then show that two elements (x_0, β_0) and (x_1, β_1) in $(x : A) \times \mathsf{Path}\ B\ y\ (f\ x)$ are path-connected. This is obtained by the definitions

$$\begin{array}{rcl} \omega_0 & = & \mathsf{comp}^i\ E(i/1-i)\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ x_0]\ (\beta_0\ j) \\ \omega_1 & = & \mathsf{comp}^i\ E(i/1-i)\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ x_1]\ (\beta_1\ j) \\ \theta_0 & = & \mathsf{fill}^i\ E(i/1-i)\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ x_0]\ (\beta_0\ j) \\ \theta_1 & = & \mathsf{fill}^i\ E(i/1-i)\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ x_1]\ (\beta_1\ j) \\ \omega & = & \mathsf{comp}^j\ A\ [(k = 0) \mapsto \omega_0, (k = 1) \mapsto \omega_1]\ (g\ y) \\ \theta & = & \mathsf{fill}^j\ A\ [(k = 0) \mapsto \omega_0, (k = 1) \mapsto \omega_1]\ (g\ y) \end{array}$$

  so that we have $\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \theta_0 : E$ and $\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \theta_1 : E$ and $\Gamma, j : \mathbb{I}, k : \mathbb{I} \vdash \theta : A$. If we define

$$\delta = \mathsf{comp}^i\ E\ [(j = 0) \mapsto v\ y, (j = 1) \mapsto u\ \omega, (k = 0) \mapsto \theta_0, (k = 1) \mapsto \theta_1]\ \theta$$

  we then have

$$\langle k \rangle\ (\omega, \langle j \rangle \delta) : \mathsf{Path}\ ((x : A) \times \mathsf{Path}\ B\ y\ (f\ x))\ (x_0, \beta_0)\ (x_1, \beta_1)$$

  as desired. We have hence shown that f is an equivalence, so we have constructed $\mathsf{equiv}^i\ E : \mathsf{Equiv}\ A\ B$.

  Using this we can now define the composition for the universe:

$$\Gamma \vdash \mathsf{comp}^i\ C\ [\varphi \mapsto E]\ A_0 = \mathsf{Glue}\ [\varphi \mapsto (E(i1), \mathsf{equiv}^i\ E(i/1-i))]\ A_0 : C(i1)[\varphi \mapsto E(i1)]$$