ALGEBRAIC INTEGRATION THEORY

THIERRY COQUAND

Abstract. We give a constructive treatment of Segal's notion of integration algebra.

Introduction

In [1] we give a constructive treatment of two fundamental notes of Stone. By unexpected chance, it turns out that this applies directly to Segal's notion of integration algebra.

1. Integration Algebra

An *integration algebra* is a pair (R, E) where R is a $\mathbb{Q}$-algebra with unit and E a linear functional on A such that

(1) $E(A^2) \geq 0$ and $= 0$ iff $A = 0$
(2) for any element B there exists b such that $E(BA^2) \leq bE(A^2)$ for all $A \in R$
(3) $E(1) = 1$

Let (R, E) be an integration algebra. We write $(A, B) = E(AB)$ for $A, B \in R$. We can think now of R as a preHilbert space and each element of R defines a bounded self-adjoint operator on this space.

We define an order $B \leq C$ on R by

$$(BA, A) \leq (CA, A)$$

for all $A \in R$.

We have clearly $0 \leq A^2$ for all A, and more generally $0 \leq BA^2$ if $0 \leq B$. What is remarkable is the following result.

Proposition 1.1. *If $0 \leq B$ and $0 \leq C$ then $0 \leq BC$*

Proof. We can assume $0 \leq C \leq 1$. We define $C_0 = C, C_{n+1} = C_n - C_n^2$ One shows $0 \leq C_{n+1} \leq C_n \leq 1$ and $C_{n+1}^2 \leq C_n^2$. Since $C = C_1^2 + \cdots + C_n^2 + C_{n+1}$ we have $C_n^2 \leq C/n+1$ and so $C_n^2 \to 0$. □

It follows that we have an archimedian ordered ring $R, \leq$ with a strong unit 1. Hence we know how to define the maximal spectrum $Max(R)$ of R which is a compact Hausdorff locale. The functional E can then be seen as a probability measure on $Max(R)$.

References

[1] Th. Coquand About Stone's Notion of Spectrum submitted
[2] I. Segal Algebraic Integration Theory Bull. Amer. Math. Soc. 71 1965 419–489.

Department of Computer Science, Chalmers University of Technology and Gothenburg University, SE-412 96 Göteborg, Sweden. E-mail: coquand@cs.chalmers.se