Type Theory and Univalent Foundation

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Clermont-Ferrand, October 17, 2013

This talk

Revisit some questions discussed by Russell at the beginning of Type Theory

-Russell's Paradox (1901)

-Theory of Descriptions (1905)

-Theory of Implication (1906)

-Extensionality (1925)

-Reducibility (1925)

-Theory of structures and similarity of structures

This talk

Univalent Foundation (Voevodsky 2009) brings new light to these questions

It also suggests a new approach to the foundation of category theory

The Theory of Implications

American Journal of Mathematics 1906

Russell shows how to build a Boolean algebra from the axioms of implication

Observe that if $p \equiv q$, q *may be substituted for* p, *or vice versa, in any formula involving no primitive ideas except implication and negation, without altering the truth or falsehood of the formula. This can be proved in each separate case, but no generally, because we have no means of specifying (with our apparatus of primitive ideas) that a complex* $C(p, q)$ *is to be one that can be built up out of implication and negation alone*

The essence of explanation of extensionality

Introduction to the second edition of Principia Mathematica

Generalized in 1925, written under the influence of Wittgenstein

A function can only enter into a proposition through is values

All functions of functions are extensional ... Consequently there is no longer any reason to distinguish between functions and classes

This assumption if fundamental in the following theory. It has its difficulties, but for the moment we ignore them. It takes the place (not quite adequatly) of the axiom of reducibility

Also discussed in *Introduction to Mathematical Philosophy*, 1919

The quantifiers are extensional functions of functions

Structures

The notion of *structure* played an important role for Russell

For instance in *Introduction to Mathematical Philosophy*, 1919

Chapter VI *Similarity of Relations*

We may say, of two similar relations, that they have the same "structure"

Definition of "relation number" and operations of these relation numbers

Also in *Human Knowledge. Its Scope and Limits*, 1948

(Influence on Tarski's definition of logical notions?)

Type Theory

Elegant formulation by A. Church (1940)

Simple types o (type of propositions) and ι (type of individuals)

$\alpha \to \beta$ (function types) written $(\beta)\alpha$ by Church

10^o Propositional extensionality $(p \equiv q) \;\to\; p = q$ (already in Russell 1925)

$10^{\alpha\beta}$ Function extensionality $(\forall x^\alpha . f\ x = g\ x) \;\to\; f = g$

9^α Axiom of Description $\forall f^{\alpha\to o}.\forall x^\alpha.\ f\ x \wedge (\forall y^\alpha . f\ y \to x = y) \;\to\; f\ (\iota\ f)$

11^α Axiom of Choice $\forall f^{\alpha\to o}.\forall x^\alpha.\ f\ x \;\to\; f\ (\iota\ f)$

Remarks

We can rewrite the extensionality axioms as

10^o Propositional extensionality $p = q \;\equiv\; (p \equiv q)$

$10^{\alpha\beta}$ Function extensionality $f = g \;\equiv\; (\forall x^\alpha . f\ x = g\ x)$

The axioms $1 - 6$ are about basic laws of logic

The axioms $7 - 8$ are about individuals (axiom of infinity)

Church introduced type of functions not necessarily proposition valued

E.g. $\iota \to \iota$ if ι primitive type of individuals

Axiom of Reducibility

In Church's formulation this becomes the fact that we have a quantification

$$\forall : (\alpha \rightarrow o) \rightarrow o$$

where α is *any type*, which can be more complex than o

In Russell 1925 system, one would have a stratification o_n of propositions, e.g.

$$\forall : (o_1 \rightarrow o_1) \rightarrow o_2$$

Propositions as Types

The next step occurs in the 70s through the work of Curry, Howard, de Bruijn, Tait, Scott, Martin-Löf, Girard, $\ldots$

In *natural deduction* the laws for proving a proposition are the same as the laws for building an element of a given type

E.g. $\lambda x.t$ is of type $A \rightarrow B$ if t is of type B given x of type A

$c\ u$ is of type B if c is of type $A \rightarrow B$ and u of type A

It is natural to identify *propositions* and *types*

Propositions as Types

So the type of propositions can be thought of as a type of (small) types

Universal quantification corresponds to an operation

$(\Pi x : A)B$ if $B(x)$ is a *dependent type* over $x : A$

E.g. $\lambda x.t$ is of type $(\Pi x : A)B$ if t is of type B given x of type A

Propositions as Types

Unification: the laws of logic and term formation are the *same*

Girard's Paradox (1971): if there is a type of all types we have a contradiction

The type of proposition can be thought of as a "universe" of (small) types

Russell's stratification is similar to the stratification of Grothendieck's universes

Explanation of the (intuitionistic version of the) axioms $1 - 6$

Propositions as Types

Howard, Scott and then Martin-Löf introduced a *new* logical quantification

$(a, b) : (\Sigma x : A)B$ if $a : A$ and $b : B(a)$

$\pi_1(z) : A$ if $z : (\Sigma x : A)B$

$\pi_2(z) : B(\pi_1(z))$ if $z : (\Sigma x : A)B$

This is a "constructive" and explicit version of existential quantification

Equality?

What should be the rules of equality in this new interpretation?

In particular, should the equality be extensional?

Martin-Löf (1973): we should have

$\mathsf{refl}\ a : \mathsf{Id}_A\ a\ a$

$J(d,p) : C(x,p)$ if $C(x,p)$ type $(x : A, p : \mathsf{Id}_A\ a\ x)$ and $d : C(a, \mathsf{refl}\ a)$

Furthermore $J(a, \mathsf{refl}\ a) = d : C(a, \mathsf{refl}\ a)$

This was introduced purely "formally", based upon "symmetry reason"

Equality?

In particular we have

$C(a)$ implies $C(x)$ if $p : \mathsf{Id}_A\ a\ x$

Any property of a holds for x if $\mathsf{Id}_A\ a\ x$

This is Leibnitz' law of indiscernability of identicals

The new (until 1973) property is

Any element in $(\Sigma x : A)\mathsf{Id}_A\ a\ x$ is *equal* to $(a, \mathsf{refl}\ a)$

Equality?

With this formulation, function extensionality does not hold

What should be the equality for the universe(s)?

When are two small types equal?

Univalent Foundation

Voevodsky (2009) introduced the following hierachy

A type A is contractible iff $(\Sigma a : A)(\Pi x : A)\mathsf{Id}_A\ a\ x$ holds (is inhabited)

A type A is of *hlevel* 0 iff it is contractible

A type A is of hlevel $n + 1$ iff all $\mathsf{Id}_A\ a_0\ a_1$ are of hlevel n

In particular

A *proposition* is a type of hlevel 1

A *set* is a type of hlevel 2

A *groupoid* is a type of hlevel 3

Univalent Foundation

contr A is $(\Sigma a : A)(\Pi x : A)\mathsf{Id}_A\ a\ x$

prop A is $(\Pi a_0 : A)(\Pi a_1 : A)\mathsf{contr}\ (\mathsf{Id}_A\ a_0\ a_1)$

set A is $(\Pi a_0 : A)(\Pi a_1 : A)\mathsf{prop}\ (\mathsf{Id}_A\ a_0\ a_1)$

groupoid A is $(\Pi a_0 : A)(\Pi a_1 : A)\mathsf{set}\ (\mathsf{Id}_A\ a_0\ a_1)$

One can show that prop A holds iff $(\Pi a_0 : A)(\Pi a_1 : A)\ \mathsf{Id}_A\ a_0\ a_1$

Univalent Foundation

The type $(\Sigma x : A)\mathsf{Id}_A\ a\ x$ is *always* contractible for any A and $a : A$

Indeed any element (x, p) of this type is equal to $(a, \mathsf{refl}\ a)$

Univalent Foundation

A map $f : A \to B$ is an *equivalence* iff all fibers of f are contractible

$\mathsf{equiv}\ f$ is $(\Pi y : B)\mathsf{contr}\ (\Sigma x : A)\mathsf{Id}_B\ (f\ x)\ y$

Write $A \simeq B$ for stating that there exists an equivalence between A and B

E.g. if A and B are sets we get back the notion of *bijection* between sets

If A and B are groupoids notion of *categorical equivalence* between groupoids

If A and B are propositions notion of *logical equivalence* between propositions

Univalent Foundation

The identity map $A \to A$ is always an equivalence

We have a map $Id_U\ A\ B \to A \simeq B$

The Axiom of Univalence claims that this map is an equivalence

$$A = B \simeq (A \simeq B)$$

Univalent Foundation

This generalizes the axiom

$$p = q \equiv (p \equiv q)$$

This *implies* function extensionality

Univalent Foundation

Function extensionality is equivalent to

a product of contractible types is contractible

This implies that

$(\Pi x : A)B$ is a *proposition* if $B(x)$ is a family of propositions over A

This holds for *any* type A

Structures

We can define structures using universes and dependent sum types, e.g.

$S = (\Sigma X : U)\ (X \to X) \times X \times \mathsf{set}\ X$

is the type of structures $s = (X, f, a, p)$ with $f : X \to X$ and $a : X$

$p : \mathsf{set}\ X$ is a proof that X is a set

We can define $\mathsf{Iso}_S\ s_0\ s_1$ as usual

Corollary of the Axiom of Univalence: two *isomorphic* elements of S are *equal*

More precisely, the canonical map $\mathsf{Id}_S\ s_0\ s_1 \to \mathsf{Iso}_S\ s_0\ s_1$ is an equivalence

Structures

This implies that two *isomorphic* structures have the same *properties*

This is not true in a set theoretic framework, e.g.

$s_0 = (\mathbb{N}, +1, 0)$

$s_1 = (\mathbb{N} - \{0\}, +1, 1)$

are isomorphic, but 0 is in the carrier of s_0 and not in the one of s_1

When two relations have the same structure, their logical properties are identical, except such as depend upon the membership of their fields

Russell (1959) *My philosophical development*

Structures

Another example of a structure is the notion of *poset*

-A set A

-A binary relation $R(x, y)$ over A which is *proposition* valued

-It is reflexive and transitive

-We have $x =_A y \ = \ (R(x, y) \times R(y, x))$

Structures

These are usual "algebraic" mathematical structures, at the *set* level

A *category* is a structure at the *groupoid* level

This is the hlevel 3 version of a poset

Structures

A category is given by

-A type A of hlevel 3

-A binary relation $R(x, y)$ over A which is *set* valued

-It is reflexive and transitive, with associativity and neutral element

-We define $\mathsf{Iso}_R(x, y)$, which is a *set*

-We have $x =_A y \ = \ \mathsf{Iso}_R(x, y)$

E.g. we can define the category of sets, the category of groups, $\ldots$

Structures

The notion of *groupoid* is more fundamental than the notion of *category*

To define a category we have *first* to introduce its collection of objects

This collection is

not a *set*, type of hlevel 2

but a *groupoid*, type of hlevel 3

Axiom of Description

It is natural to introduce a new modality operator

inhabited A

This is a *proposition* expressing that A is inhabited

The axiom is that

inhabited $A \;\rightarrow\;$ prop $B \rightarrow (A \rightarrow B) \rightarrow B$

Given this modality, we can *define* the existential quantifier

$(\exists x : A)B$ defined as inhabited $(\Sigma x : A)B$

Axiom of Description

In this setting the Axiom of Description is provable

$(\exists x : A)B \rightarrow (\Sigma x : A)B$

as soon as A is a *set* and that there is at most one $x : A$ satisfying B

Indeed, in this case $(\Sigma x : A)B$ is a *proposition*

This gives a *new* analysis of the description operator

Requires as an argument a *proof* that there exists exactly one witness

$(\exists! x : A)B$ is $(\exists x : A)(B(x) \times (\Pi u : A)B(u) \rightarrow \mathsf{Id}_A\ x\ u)$

$(\exists! x : A)B \ \rightarrow (\Sigma x : A)B$

Axiom of Choice

The Axiom of Global Choice in the form

$(\Pi X : U)\ (\mathsf{set}\ X) \rightarrow (\mathsf{inhabited}\ X) \rightarrow X$

is provably false

No "invariant" global choice function

Reducibility

Voevodsky also suggested the following version of the reducibility axiom

We have a hierarchy of universe $U_0,\ U_1,\ U_2,\ \ldots$

The new axiom, "resizing" axiom, states that any proposition is in U_0

For instance

$(\Sigma X : U_2)\ \mathsf{Id}_{U_2}\ U_1\ X$

is in U_0

Connection between "size" and complexity of equality

Constructive Type Theory

Bishop defined a *set* as a *collection*

together with a relation which is an *equivalence* relation

A *type* is interpreted as a *cubical collection* such that

any open box can be filled

This generalizes the notion of equivalence relation

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