HOW TO DEFINE MEASURE OF BOREL SETS

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Abstract. One of the first definitions of measure, by Borel, provides an early example of *generalised recursive definition* of a function. It presented however an ambiguity problem. Lebesgue's work provides a solution to this problem, but a natural question [10] is if this problem has not a purely inductive solution. The goal of this paper is to show that this problem has a solution using some concepts of functional analysis essentially due to F. Riesz [13]. Furthermore, this solution is not only purely inductive, but is expressed naturally using only intuitionistic logic. This work can also be seen as a possible approach to constructive probability theory.

Introduction

One of the first definitions of measure, by Borel, presented a subtle ambiguity problem. We recall here Borel's definition, and in order to simplify the presentation, we consider only subsets of the open interval $(0,1)$. This definition is an early example of a *generalised inductive definition* and of a *generalised recursive definition* of a function.

- An open interval (r,s) is a well-defined set[1] and its measure $\mu((r,s))$ is $s-r$.
- If we have a disjoint collection of well-defined sets A_n its union A is a well-defined set, and $\mu(A) = \Sigma\mu(A_n)$.
- Finally, if $A \subseteq B$ are two well-defined sets, then the difference $B - A$ is a well-defined set, and $\mu(B-A) = \mu(B) - \mu(A)$.

The problem in this definition of the measure is that it may depend in a non-extensional way on the presentation of a well-defined set: it is not clear a priori that if A and B denote the *same* set, then we have $\mu(A) = \mu(B)$. In the first edition of his book [3], Borel limits himself to prove what became known as Heine-Borel's theorem and states in a footnote that the ambiguity in his definition of measure can be solved using ideas similar to the ones contained in this proof. In later editions, Borel sketches a proof in an appendix that any well-defined set can be approximated in a suitable sense by a finite union of open intervals [3], and in this way solves the ambiguity in his definition of measure[2].

It is a natural question however if "Borel's measure problem", as it was called in Lusin's book [10], has not a purely inductive solution. The goal of this paper is to show that this problem has a natural solution using some concepts essentially due to F. Riesz [13].

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[1] In the first edition of [3] such subsets were called measurable and then, after Lebesgue's work, Borel called them *well-defined* subsets.

[2] This addition came after Lebesgue's work on measure theory.

Furthermore, this solution is not only purely inductive, but is naturally expressed using only intuitionistic logic.

In order to simplify the problem, we shall not consider measure for Borel subsets of $(0, 1)$ but for Borel subsets of Cantor space. In [12] is presented a constructive definition of Borel subsets of Cantor space Ω and their inclusion relation. They are seen as symbolic expressions built from the Boolean algebra of simple (closed and open) subsets of Ω by formal countable disjunction and conjunction. If X is such a symbolic expression, it is clear for instance how to define by induction the formal complement X' of X. Classically one can think of symbolic expressions as sets of points, and define the inclusion relation extensionally. Constructively, it is still possible to define $X \subseteq Y$ for X, Y Borel subsets without mentioning points, and this is done in [12] using a suitable infinitary one-sided sequent calculus. Using this approach the law $\Omega \subseteq X \cup X'$ for instance can be justified constructively.

A theory of Lebesgue measure on Ω is also presented in [12], starting from a measure $\mu(b) \in [0, 1]$ of simple (closed and open) subsets. Constructively, if we want the measure of a subset to be a computable real, we cannot define in general the measure of even open sets, and the question of measure of Borel subsets is usually not addressed.

Here we define the measure of a Borel set as a bounded "hyperarithmetical real", i.e. a real built from rationals by repeated (may be transfinitely) sups and infs of bounded sequences. The starting point is to follow Borel [3] and try to define $\mu(X)$ by induction on the construction of X. One insight is that it is more elegant to define by induction the function

$$\mu_X : b \longmapsto \mu(X \cap b)$$

where b denotes an arbitrary simple set of Cantor space Ω. Indeed, it will be shown that μ_X can be defined by *structural* induction on X, that is, if X_n are the components of X, μ_X is a function of μ_{X_n} while $\mu(X)$ is not a priori a function of $\mu(X_n)$.

This development is done in an intuitionistic framework using as primitive the notion of generalised inductive definition. This work can also be seen as a possible approach to constructive probability theory, and we illustrate this in the last section with a presentation of Borel's normal number theorem.

We formulate this approach in the framework of the theory of Riesz spaces [20]. This theory, which originated from Riesz's work [13] and Stone's papers [15, 16], is well suited for a constructive and point-free development of measure theory. Actually there is a great analogy between some motivations in point-free topology [9] and motivations in the theory of Riesz spaces where people strive for elementary and "representation free" development (cf. the preface of [20]). We then apply this analysis to the case of a measure in a monotone σ-complete ordered space [19].

1. Inductive Definition of Borel sets

The following fact, proved in [8] and recalled in [7] will play a key role here. We are going to prove a similar result for the space of bounded Baire functions on Ω. Let B the Boolean algebra of simple subsets of Ω.

Theorem 1.1. *The σ-algebra of Borel subsets of Cantor space Ω is the σ-completion B_1 of B.*

We recall the definition of the σ-completion of B. We have first a Boolean algebra map $i : B \to B_1$. Also B_1 is σ-complete, and for any other map $f : B \to A$ in a σ-complete Boolean algebra A there exists a unique σ-algebra map $\bar{f} : B_1 \to A$ such that $\bar{f} \circ i = f$.

Hence in order to define the measure of Borel sets of Cantor space, it is enough to extend the finitely additive measure on B to a countably additive measure on its σ-completion B_1.

In [12] an explicit construction of B_1 is given. It follows for instance from this construction that i is one-to-one, fact which can also be proved by using the σ-complete algebra construction described in [6].

2. Riesz Spaces

2.1. **Algebraic Theory.** We define an *ordered space* to be a vector space E on the rationals, with an order relation $\leq$ such that

$$x \leq y \to x + z \leq y + z$$

An ordered space is a *Riesz space* iff any two elements x, y have a least upper bound $x \vee y$. A natural other axiom would be that $r \geq 0$ then $x \leq y$ implies $rx \leq ry$; however, as noticed in [2], this is, rather surprisingly, actually provable from the only axiom that we have an ordered space which is a lattice.

We shall work only with Riesz spaces that have a *strong unit*: we assume given a distinguished element $1 \in E$ such that for any $x \in E$ there exists n such that $x \in [-n, n]$ that is $-n \cdot 1 \leq x \leq n \cdot 1 \in E$. If r is a rational we shall write $r \in E$ for $r \cdot 1 \in E$.

Given two Riesz spaces E_1 and E_2, a *Riesz space map* will be a linear map $f : E_1 \to E_2$ preserving binary sups and such that $f(1) = 1$.

In any Riesz space any two elements x, y have a greatest lower bound $x \wedge y = x + y - x \vee y$. Furthermore $E, \vee, \wedge$ is a distributive lattice. This follows from the more general result.

Theorem 2.1. *The following distributivity laws hold*

$$x \wedge \bigvee y_i = \bigvee (x \wedge y_i) \qquad x \vee \bigwedge y_i = \bigwedge (x \vee y_i)$$

whenever the corresponding bounds exist.

Proof. We show the first distributivity law. Let $y = \bigvee y_i$. We have $x \wedge y_i \leq x \wedge y$ for all n and hence $\bigvee (x \wedge y_i) \leq x \wedge y$. Conversely let z be such that $x \wedge y_i \leq z$ for all n. We have then

$$x + y_i - x \vee y_i \leq z$$

and hence

$$y_i \leq z - x + x \vee y_i \leq z - x + x \vee y$$

It follows that we have

$$y \leq z - x + x \vee y$$

and thus $x \wedge y \leq z$ as desired. The proof of the other distributive law is similar. □

Corollary 2.2. *For any Riesz space*

$$\{x \in [0,1] \mid x \wedge (1-x) = 0\}$$

is a Boolean algebra, written $G(E)$, called the Boolean algebra of components *of E.*

We define as usual $x^+ = x \vee 0$, $x^- = (-x) \vee 0$. We have $x = x^+ - x^-$.

An element x is called positive iff $0 \leq x$. An important relation on positive elements is the orthogonality relation: $x \perp y$ iff $x \wedge y = 0$.

Theorem 2.3. *If $x \perp y_1$ and $x \perp y_2$ then $x \perp y_1 + y_2$. If $x \perp y$ and $z \leq y$ then $x \perp z$. We always have $x^+ \perp x^-$ and if $a \perp 1$ then $a = 0$.*

As noted by Dieudonne, this theorem goes back to Euclides, since it can be seen as a general form of results about numbers that are relatively prime.

An f-ring or *function ring* [2] is a Riesz space which is also a commutative ordered ring such that $a(b \wedge c) = ab \wedge ac$ if $a \geq 0$.

2.2. **Dedekind σ-complete Riesz Spaces.** A Riesz space is called *Dedekind σ-complete* iff any sequence x_n which is bounded above has a least upper bound $\vee x_n$. This notion will play a key role here.

In a Dedekind σ-complete Riesz space, any sequence x_n which is bounded below has a greatest lower bound $\wedge x_n$. Indeed, if $z \leq x_n$ for all n we have that $z - \bigvee(z - x_n)$ is a inf of the sequence x_n. It follows from theorem 2.1 that the Boolean algebra of components of a Dedekind σ-complete Riesz space is a σ-complete Boolean algebra.

2.3. **Dedekind σ-completion of Riesz Spaces.** A map $f : E_1 \to E_2$ of Riesz spaces is a σ-map iff it preserves sups of sequences that are bounded above. Let E be a Riesz space. We say that $E_1, i : E \to E_1$ is a *σ-completion* of E iff E_1 is Dedekind σ-complete, and for any other map $f : E \to F$ of E in a Dedekind σ-complete Riesz space F there exists one and only one σ-map $\bar{f} : E_1 \to F$ such that $\bar{f} \circ i = f$.

The existence and uniqueness of the σ-completion of a Riesz space can be seen abstractly. We will give later an explicit construction. The importance of this construction is that it gives a point-free way of defining bounded Baire functions over a space. This is justified by the following result.

Theorem 2.4. *Let X be a compact Hausdorff space, and $C(X)$ the Riesz space of continuous function over X, with the constant function 1 as strong unit. The Riesz space of bounded Baire functions on X is the σ-completion of $C(X)$. The corresponding σ-complete Boolean algebra of its components is the Boolean algebra of Baire subsets of X if X is separable.*

In general, $C(X)$ is Dedekind σ-complete iff X is the representative space of a σ-complete Boolean algebra [17].

Proof. If X is a compact Hausdorff space, let $B(X)$ be the Riesz space of bounded Baire functions on X, that is the space of functions we get by repeated sups and infs of bounded sequences of functions, starting from the continuous functions.

We shall make use of the following results of [15] (we shall prove later a representation-free version of these results). First, any Dedekind σ-complete Riesz space is of the form $C(Y)$, where Y is the representative space of a σ-complete Boolean algebra. It is also proved in [17] that, in such a case, any bounded Baire function f in $B(Y)$ determines a unique $\phi(f) \in C(Y)$ such that $f(y) = \phi(f)(y)$ except on a meager set.

We can now show that $B(X)$ is the σ-completion of $C(X)$. Any map $\psi : C(X) \to C(Y)$ corresponds to a continuous map $s : Y \to X$, and hence extends to a map $B(X) \to B(Y)$. If Y is the representative space of a σ-complete Boolean algebra, we compose this map with $\phi : B(Y) \to C(Y)$, and we get an extension $\bar{\psi} : B(X) \to C(Y)$ of ψ, which is uniquely determined. □

2.4. **Riesz Spaces and Boolean Algebras.** To any Boolean algebra B we associate the function ring $V(B)$ of formal finite step functions on B. This can be seen as the Q-algebra generated by symbols $v(b)$, $b \in B$ with the relations

$$vb_1 + vb_2 = v(b_1 \wedge b_2) + v(b_1 \vee b_2), \quad v1 = 1, \quad v0 = 0$$

We define $0 \leq a$ iff a can be written $a = \Sigma r_i v b_i$ with $r_i \geq 0$. The multiplication is defined by $v(b_1)v(b_2) = v(b_1 \wedge b_2)$.

We have the following universal property of the Riesz space $V(B)$ and the map $v : B \to V(B)$. Let a *valuation* on B be a map $m : B \to E$ into a Riesz space such that $m(0) = 0$, $m(1) = 1$ and

$$m(b_1) + m(b_2) = m(b_1 \vee b_2) + m(b_1 \wedge b_2)$$

Theorem 2.5. *$v : B \to V(B)$ is a universal valuation: if $m : B \to E$ is any valuation, there exists a unique map $I : V(B) \to E$ such that $I \circ v = m$.*

The importance of this construction is stressed in the work of Rota [14]. The elements of $V(B)$ may be thought of as simple random variables on a probability space, while I is the expectation operator.

2.5. **Dedekind σ-complete Riesz Spaces.** We now give a point-free version of some results of [17], and present the spectral decomposition of an element of a Dedekind σ-complete Riesz space.

To any σ-complete Boolean algebra B we associate a Riesz space $F(B)$[3] (which is actually also a function ring). An element of $F(B)$ is a family $\phi(r) \in B$ indexed by rationals such that

- there exists N such that $\phi(r) = 0$ if $r \leq -N$ and $\phi(r) = 1$ if $N < r$
- $\phi(r) = \bigvee_{s<r} \phi(s)$

Given two such elements ϕ and ψ the addition is given by the convolution product

$$(\phi + \psi)(t) = \bigvee_r [\phi(r) \wedge \psi(t - r)]$$

[3]It can be seen in [17] that with our definition, an element of $F(B)$ corresponds to a continuous function on the representative space of B.

and we define $\phi \leq \psi$ to mean $\psi(r) \leq \phi(r)$ for all r. Finally we define

$$(\phi \vee \psi)(t) = \phi(t) \wedge \psi(t)$$

Let us consider now the category **B** of Boolean σ-algebras (with σ-additive maps), and the category **R** of Dedekind σ-complete Riesz space. We have a functor $F : \mathbf{B} \to \mathbf{R}$ and conversely we have a functor $G : \mathbf{R} \to \mathbf{B}$ that to any object in **R** associates its σ-complete Boolean algebra of components.

Theorem 2.6. *$F(B)$ is a Dedekind σ-complete Riesz space, whose Boolean algebra of components is isomorphic to B.*

Proof. If $b \in B$ one defines the family $\phi_b \in F(B)$ by

$$\phi_b(r) = 0 \quad if\, r \leq 0, \quad \phi_b(r) = 1 - b \quad if\, 0 < r \leq 1, \quad \phi_b(r) = 1 \quad if\, 1 < r$$

This defines a map $\epsilon : B \to G(F(B))$ and it can then be checked directly that this map is an isomorphism. □

Theorem 2.7. *G is a right adjoint of F: we have a canonical isomorphism between the set of maps $B \to G(E)$ and $F(B) \to E$.*

Proof. We show how to define the counit of the adjunction $\tau : F(G(E)) \to E$. We suppose given a family $\phi \in G(E)$ such that

- there exists N such that $\phi(r) = 0$ if $r \leq -N$ and $\phi(r) = 1$ if $N < r$
- $\phi(r) = \bigvee_{s<r} \phi(s)$

We consider then rational partitions π of $[-N, N]$, $\pi = -N = \alpha_0 < \cdots < \alpha_n = N$ and we associate

$$t(\pi) = \Sigma \alpha_{k-1}(\phi(\alpha_k) - \phi(\alpha_{k-1}))$$

$$s(\pi) = \Sigma \alpha_k(\phi(\alpha_k) - \phi(\alpha_{k-1}))$$

We have clearly $-N \leq t(\pi) \leq s(\pi) \leq N$ and $s(\pi) - t(\pi)$ is bounded by the mesh of the partition π. Then $sup_\pi t(\pi) = inf_\pi s(\pi)$ defines an element in E. A suggestive notation for this element is

$$\int \alpha \; d\phi(\alpha) = sup_\pi t(\pi) = inf_\pi s(\pi)$$

and the map $\tau : \phi \to \int \alpha \; d\phi(\alpha)$ is the unique σ-map such that $\tau(\phi_b) = b$ if $b \in G(E)$. □

Corollary 2.8. *Let H be the free σ-complete Boolean algebra. Then $F(H)$ is the free Dedekind σ-complete Riesz space. An element of $F(H)$ is called a* bounded hyperarithmetical real.

Intuitively such a real is one that can be obtained from rationals by repeated sups and infs of bounded sequences.

We know already, by theorem 2.6, that the unit map $\epsilon : B \to G(F(B))$ is an isomorphism. We will show later that the counit map is an isomorphism as well.

3. The Measure Problem

3.1. **The Problem.** We start from the Boolean algebra of simple subsets of Cantor spaces Ω with the usual Lebesgue measure $\mu : B \to [0,1]$ which is a finitely additive map: if b_1, b_2 are disjoint then $\mu(b_1 \vee b_2) = \mu(b_1) + \mu(b_2)$.

The problem is now to extend μ to a measure μ_1 on the σ-completion B_1 of B which is σ-additive. One way to express this is that μ_1 has to be a finitely additive measure which extends μ, that is $\mu_1 \circ i = \mu$ and has to be σ-continuous: if x_n is an increasing sequence in B_1 then $\mu(\bigvee x_n) = \bigvee \mu(x_n)$.

We fix now R an arbitrary Dedekind σ-complete Riesz space and we are going to define such an extension $\mu_1 : B_1 \to R$. An alternative description of μ is that we give a linear map $I : V(B) \to R$ such that $I(1) = 1$ and $I(f) \geq 0$ if $f \geq 0$. We define

$$I(\Sigma r_i v(b_i)) = \Sigma r_i \mu(b_i)$$

Let V_1 be the σ-completion of $V(B)$. We notice first the following result, consequence of theorems 2.6 and 2.7.

Theorem 3.1. *V_1 and $F(B_1)$ are isomorphic, and so are $G(V_1)$ and B_1.*

We say that a linear map between ordered vector spaces is σ-continuous iff it preserves existing sups of infinite sequences. We are going to extend the linear map I to a σ-continuous map $I_1 : V_1 \to R$. Intuitively this defines the integral of any bounded Baire function on the representative space of B.

In particular we can take for R the set of bounded hyperarithmetical reals, i.e. the σ-completion of the rationals and we get a definition of the measure of Borel sets of Cantor space as bounded hyperarithmetical reals.

3.2. **Space of I-Bounded Measures.** On the set of linear, equivalently additive, maps $l : V(B) \to R$, we consider the ordering $l_1 \leq l_2$ iff $l_1(f) \leq l_2(f)$ whenever $f \geq 0$. The subset M_I of all linear map l such that there exists n such that $l \in [-nI, nI]$ is called the space of I-bounded measures.

Any element $f \in V(B)$ defines a I-bounded measure $\phi(f) \in M$ by taking $\phi(f)(g) = I(fg)$.

Lemma 3.2. *In a Riesz space, if $\Sigma f_i = \Sigma g_j$ for positive f_i, g_j then there exist positive h_{ij} such that $f_i = \Sigma_j h_{ij}$ and $g_j = \Sigma_i h_{ij}$.*

Proof. See for instance [4]. □

Theorem 3.3. *The ordered vector space M is a Dedekind σ-complete Riesz space. Furthermore the map $\phi : V(B) \to M$ is a map of Riesz spaces.*

Proof. The main idea in the proof is due to F. Riesz [13] and consists in the following definition of the sup $\nu = \nu_1 \vee \nu_2$. For $f \geq 0$ we let $\nu(f)$ be the sup of the countable family of reals

$$\nu_1(f_1) + \nu_2(f_2)$$

over all possible positive decompositions $f = f_1 + f_2$. This family is countable because B is countable and decidable. To show the additivity of ν assume $h = f + g$. It is clear that we have $\nu(f) + \nu(g) \leq \nu(h)$ since any decomposition of f and of g will give one decomposition of h. To show the converse inequality $\nu(h) \leq \nu(f) + \nu(g)$ one takes an arbitrary decomposition of h and uses lemma 3.2. □

Let now V_1 be the σ-completion of $V(B)$ and $\psi : M_I \to R,\ l \longmapsto l(1)$ be the evaluation map, which is clearly σ-continuous.

Corollary 3.4. *There exists a unique σ-map $\phi_1 : V_1 \to M$ extending ϕ. The composition $I_1 = \psi \circ \phi_1 : V_1 \to R$ is a σ-continuous extension of I to V_1.*

It would be possible to generalise this construction starting from an arbitrary function ring (not necessarily of the form $V(B)$).

4. Extension to monotone σ-complete spaces

We now extend our analysis to the case of a valuation $\mu : B \to R$ where B is an arbitrary boolean algebra, not necessarily countable or decidable, and R is an ordered vector space, with a strong unit 1, which is not any more supposed to be a lattice. We assume instead that R is *monotone σ-complete*: if x_n is a bounded monotone sequence then $\bigvee x_n$ exists. We let $I : V(B) \to R$ be the corresponding map, and V_1 be the σ-completion of $V(B)$. The problem is to extend the map I to a σ-continuous map $V_1 \to R$ [4].

Lemma 4.1. *A monotone σ-complete Riesz space is Dedekind σ-complete.*

We let M_I be the space of all I-bounded maps $l : V(B) \to R$. It is direct that M is also monotone σ-complete for the ordering: $l_1 \leq l_2$ iff $l_1(f) \leq l_2(f)$. However M_I is not necessarily a Riesz space any more and we have to proceed in a different way in order to extend I to V_1. We start by noticing that we still have a map $\phi : V(B) \to M$ defined by $\phi(f)(g) = I(fg)$.

Lemma 4.2. *In M, the sup of $\phi(f_1)$ and $\phi(f_2)$ exists and is $\phi(f_1 \vee f_2)$.*

Proof. Let I_a be $\phi(v(a))$. We limit ourselves to show that we have

$$I_{a_1 \vee a_2} = I_{a_1} \vee I_{a_2}$$

Indeed, it is clear that $I_{a_i} \leq I_{a_1 \vee a_2}$. Suppose $I_{a_i} \leq l \in M$. We can write, for $f \in V(B)$

$$fv(a_1 \vee a_2) = f_1 + f_2$$

with $f_i v(a_i) = f_i$. We have then

$$I_{a_1 \vee a_2}(f) = I_{a_1}(f_1) + I_{a_2}(f_2) \leq l(f_1) + l(f_2) = l(f)$$

as desired. □

[4] It is noted in [19]: "The difficulties arising in dropping the lattice condition on R seem analogous to the difficulties encountered in the theory of C^*-algebras when going from the commutative to the non-commutative case. Indeed, the self-adjoint part of a C^*-algebra is a partially ordered vector space which is a lattice if, and only if, the algebra is commutative."

This shows that the image of ϕ in M_I determines a Riesz subspace of M_I.

Lemma 4.3. *If $y \leq x$ and $x \vee z$, $y \vee z$ exist then*

$$0 \leq x \vee z - y \vee z \leq x - y$$

Proof. Indeed, we have that $x \wedge z, y \wedge z$ exist and that

$$x \vee z = x + z - x \wedge z, \quad y \vee z = y + z - y \wedge z$$

and so

$$x \vee z - y \vee z = x - y + y \wedge z - x \wedge z \leq x - y$$

□

The next crucial lemma is a form of minmax principle valid in *any* monotone σ-complete space E.

Lemma 4.4. *If a_n, b_m are bounded sequences, a_n increasing and b_m decreasing, and $a = \bigvee a_n$, $b = \bigwedge b_m$ and $a_n \vee b_m$ exists for all n, m then*

$$\bigwedge_m \bigvee_n (a_n \vee b_m) = \bigvee_n \bigwedge_m (a_n \vee b_m)$$

and the common value is $a \vee b$.

Proof. We let u_n be $\bigwedge_m (a_n \vee b_m)$ while v_m is $\bigvee_n (a_n \vee b_m)$. It is clear that u_n is increasing, v_m is decreasing and $u_n \leq a_n \vee b_m \leq v_m$ for all n, m. Let u be $\bigvee u_n$ and v be $\bigwedge v_m$. We have

$$v - u = \bigwedge_{n,m} [v_m - u_n]$$

Also, if $p \geq n$, $q \geq m$, using lemma 4.3

$$a_p \vee b_m - a_n \vee b_q \leq a_p - a_n + b_m - b_q \leq a - a_n + b_m - b$$

and hence

$$v_m - u_n = \bigvee_{p,q} [a_p \vee b_m - a_n \vee b_q] \leq a - a_n + b_m - b$$

Since $\bigwedge_n (a - a_n) = \bigwedge_m (b_m - b) = 0$ it follows that we have $\bigwedge_{n,m} (v_m - u_n) = 0$. Hence $v = u$.

We have clearly $a \leq v$ and $b \leq v$. Also, if $a \leq c$ and $b \leq c$ we have $u \leq c$. Hence $v = u$ is the sup of a and b. □

Theorem 4.5. *If F is a monotone σ-complete ordered space, and E a subspace such that any two elements in E has a sup, then the same property holds for the least monotone σ-complete ordered subspace E_1 containing E. Hence E_1 is a Dedekind σ-complete Riesz space.*

Proof. This follows directly from lemma 4.4. □

Corollary 4.6. *The map $I : V(B) \to R$ has a unique σ-continuous extension $V_1 \to R$.*

Proof. Follows from lemma 4.2 and theorem 4.5. □

5. Constructive Probability Theory

We give a simple example following the presentation of Borel's number theorem in the first chapter of [1]. We let $r_i : \Omega \to \{-1, 1\}$ be the Rademacher map

$$\omega \longmapsto 2\omega_i - 1$$

and $s_n(\omega)$ be $\Sigma_{i\leq n} r_i(\omega)$. We can define the simple sets

$$b_{n,k} = \{\omega \in \Omega \mid \quad |\frac{s_n(\omega)}{n}| \leq \frac{1}{k} \quad \}$$

'and the Borel subset

$$N = \bigwedge_k \bigvee_m \bigwedge_{n\geq m} b_{n,k}$$

In the classical approach, N is defined as a set of points. A sequence element of the set N is called *normal*. Here N is not defined as a set of points but as a symbolic expression.

It is shown in [1] how to define a family of simple sets a_n, of the form b'_{n,k_n} for a suitable increasing sequence k_n, such that $\Sigma\mu(a_n)$ converges and that, for all m, we have

$$N' \subseteq \bigvee_{n\geq m} a_n$$

In [1] this is understood as inclusion of subsets, but we can make sense of this inclusion using the calculus described in [12], that is, by using only the laws of σ-complete Boolean algebras. Indeed we have to prove

$$\bigwedge_{n\geq m} b_{n,k_n} \subseteq N$$

and this follows from the fact that k_n is increasing, so that $k \leq k_n$ for n large enough if k is fixed, and that $b_{n,k}$ is decreasing in k.

It follows from this, using the σ-additivity of the measure m, that we have

$$m(N') \leq \Sigma_{n\geq m}\mu(a_n)$$

and hence $m(N') = 0$ and $m(N) = 1$. As noticed in [1] this is quite remarkable because N', as a set of points, is not countable. Thus, this theorem requires a non trivial notion of measure.

Finally we note that this approach is reminiscent of Borel's own description of Borel subsets [3] that he called "ensembles bien définis", stressing the symbolic way in which these sets are defined rather than looking at them extensionally as sets of points.

6. Representation Theory

6.1. **Riesz Space.** Following [16] we can see any Riesz space E with a strong unit as a set of continuous functions over a compact Hausdorff space X. It may be interesting to give a direct point-free description of this space.

Theorem 6.1. *The following geometrical propositional theory with atoms elements of E*

- $a, -a \vdash$

- $a + b \vdash a, b$
- $a \vdash$ *if* $a \leq 0$
- $\vdash 1$
- $a \vee b \vdash a, b$
- $a, b \vdash a \wedge b$
- $a \vdash \bigvee_{r>0} a - r$

describes a compact regular space X. The points of this space are the models of this propositional theory: the element a may be thought of as the proposition $a(x) > 0$ where x is a generic point of X. A point of this space defines exactly one continuous map $a \longmapsto a(x)$, such that $a \geq 0$ implies $a(x) \geq 0$.

This space X can be called the spectrum of E. We will denote by $Sp(E)$ the geometrical propositional theory that defines X.

We can associate to $a \in E$ the spectral decomposition $E_\lambda = \lambda - a$. This is an open subset of X and E_{λ_1} is well-inside E_{λ_2} if $\lambda_1 < \lambda_2$. Furthermore if $a \in [-n, n]$ we have $E_\lambda = \emptyset$ if $\lambda \leq -n$ and $E_\lambda = X$ if $n < \lambda$. Finally $E_\lambda = \cup_{\mu<\lambda} E_\mu$. Thus following [17] each element of E can be seen as a continuous function $X \to R$ i.e. as an element of $C(X)$ such that

$$E_\lambda = \{x \in X \mid a(x) < \lambda\}$$

Let us define $\vdash_{fin}$ to be the entailment relation generated by the finitary rules (all but the last one). We have the following characterisation.

Theorem 6.2. *$a_1, \ldots, a_n \vdash_{fin} b_1, \ldots, b_m$ iff $1 \wedge \wedge a_i^+ \leq N(\vee b_j^+)$ for some N.*

In the special case where $n = 0$ we get that $1 \leq N(\vee b_j^+)$ for some N. If $m = 0$ we get that $\wedge a_i^+ = 0$.

Proof. We follow the method of [5], and prove that the relation

$$1 \wedge \wedge a_i^+ \leq N(\vee b_j^+) \;\; for\ some\ N$$

is an entailment relation, and that it validates all the axioms of $\vdash_{fin}$. We have to prove that, if we have for some positive $a \leq 1, x, y$ and some N

$$a \wedge x \leq Nb \quad a \leq N(b \vee x)$$

then $a \leq Mb$ for some M. But we have $a \leq Na$ and

$$Na \wedge N(b \vee x) \leq N(b \vee (a \wedge x)) \leq N(b \vee Nb) \leq N^2 b$$

so that $a \leq Mb$ with $M = N^2$. □

Corollary 6.3. *If $\vdash_{fin} a$ then there exists N such that $1 \leq Na$.*

Proof. We know that $1 \leq Na^+$ for some N by the theorem. Using theorem 2.3 we have $a^- \perp a^+$ and hence $a^- \perp Na^+$. It follows that we have $a^- \perp 1$ and hence $a^- = 0$ by theorem 2.3. Hence $a = a^+$ and $1 \leq Na$. □

Corollary 6.4. *If $\vdash a$ then there exists N such that $1 \leq Na$.*

In term of points, this says that if $a(x) > 0$ for all $x \in X$ then there exists N such that $1/N \leq a(x)$ for all $x \in X$. It follows from this that if $0 \leq a(x)$ for all $x \in X$ then $0 \leq a + 1/n$ in E for all n. Hence if 0 is the only infinitesimal in E, that is, if E is *Archimedean* [11] we have an embedding of E into $C(X)$.

It would now be possible to develop a point-free version of the Stone-Weirstrass theorem.

Lemma 6.5. *If U_j is an arbitrary covering of X it is possible to find a partition of unity $p_1, \ldots, p_n$ with $p_i \in [0,1]$ and $\Sigma p_i = 1$ and each open $p_i(x) > 0$ is well-inside one U_j.*

Proof. We first notice that $a(x) > 0$ iff $a^+(x) > 0$. This is because $a \vdash a^+$ and $a^+ = a \vee 0 \vdash a$ since $0 \vdash$. It follows that given any covering U_j we can find positive elements $a_1, \ldots, a_n$ such that $a_i(x) > 0$ is well-inside some U_j and $a_i(x) > 0$ cover X. Since $a(x) > 0$ is the union of the monotone family $a(x) > s$, $s > 0$ we can find $s_1, \ldots, s_n > 0$ such that $a_i(x) > s_i$ cover X. This means that we have in $Sp(E)$

$$\vdash \bigvee (a_i/s_i - 1)$$

and hence, by corollary 6.4, we have $1 + 1/N \leq \bigvee a_i/s_i$ in E for some N. If we define $q_i = 1 \wedge a_i/s_i \in [0,1]$ we have thus $\vee q_i = 1$. If we define next $p_i = q_i - (q_i \wedge \vee_{j<i} q_j)$, we have $p_i \in [0,1]$, each basic open $p_i(x) > 0$ is well-inside some U_j and $\Sigma_{j<i} p_j = \vee_{j<i} q_j$. In particular $\Sigma p_i = \vee q_i = 1$. ☐

Theorem 6.6. *If $f \in C(X)$ and $r > 0$ then there exists $a \in E$ such that $|f(x) - a(x)| < r$ for all $x \in X$.*

7. Spectral Decomposition

Let E be a Dedekind σ-complete Riesz space, and $B = G(E)$ its Boolean algebra of components. We know that B is σ-complete. It is now possible, using an idea going back to Riesz [13] to give a B-valued model of the propositional theory $Sp(E)$. For this we define $[\![a > 0]\!] \in E$ as

$$[\![a > 0]\!] = sup_n \ [1 \wedge na^+]$$

The sequence of elements $[\![a < r]\!] = [\![r - a > 0]\!]$ is called the *spectral decomposition* of a.

Lemma 7.1. *Any Dedekind σ-complete Riesz space is Archimedean.*

Proof. If a is a positive element of E such that $na \leq 1$ for all n, let $b = sup_n \ na$. We have $b \in [0,1]$ and $2b = sup_n \ 2na \leq b$. Hence $b = a = 0$. ☐

Theorem 7.2. *The map $a \longmapsto [\![a > 0]\!]$ defines a B-valued model of the theory $Sp(E)$. If $a \in [-n, n]$ we have $[\![a < r]\!] = 0$ if $r \leq -n$ and $[\![a < r]\!] = 1$ if $n < r$. Furthermore $[\![a < r]\!] = \bigvee_{s<r} [\![a < s]\!]$. The map $\tau : F(B) \to E$ is an isomorphism.*

Proof. We show that $u = [\![a > 0]\!]$ is a component of E. We have clearly $u \in [0,1]$. If $p = (1 - u) \wedge a^+$ we have $u + p \leq 1$ and hence $(1 \wedge na^+) + p \leq 1$ for all n. But we have also, since $p \leq a^+$

$$(1 \wedge na^+) + p \leq (n+1)a^+$$

and hence

$$(1 \wedge na^+) + p \leq u$$

for all n. It follows that we have $u + p \leq u$ and hence $p = 0$, that is $(1 - u) \perp a^+$.

It follows from this, by 2.3 that we have $(1 - u) \perp na^+$ for all n and hence, by 2.1, that we have $(1 - u) \perp u$, which means that u is a component of E.

Using lemma 7.1 one checks then $a \longmapsto [\![a > 0]\!]$ defines a $G(E)$-valued model of $Sp(E)$. It follows from this that $\phi(r) = [\![a < r]\!]$ defines an element of $F(G(E))$, which is called the *spectral decomposition* of a. One can then show [11] that we have $t(\pi) \leq a \leq s(\pi)$ for any partition π and hence

$$a = \tau(\phi) = \int \alpha \; d\phi(\alpha)$$

where τ is the map defined in theorem 2.7. □

In terms of points, if X is the space defined by $Sp(E)$, the element $[\![a > 0]\!]$ corresponds to the unique closed open subset of X which is equal to the open set $a(x) > 0$ up to a meager set [17].

Corollary 7.3. *The categories* **B** *and* **R** *are equivalent.*

It follows from this result that a Dedekind σ-complete Riesz space has always a canonical structure of function ring.

7.1. **σ-completion of a Riesz Space.** We have seen that any Riesz space E can be seen as a purely algebraic description of a compact Hausdorff space X. It is now possible to define the σ-complete Boolean algebra of Borel subsets of X as the Boolean algebra B_1 of components of the σ-completion E_1 of E, while E_1 can be seen to be the Riesz space of bounded Baire functions on X. We have also a direct description of B_1 and E_1.

Theorem 7.4. *Let B_0 be the σ-algebra generated by the symbols $a \in E$ and theory $Sp(E)$. The Riesz space $F(B_0)$ is the σ-completion of E.*

Proof. Let E_1 be any Dedekind σ-complete Riesz space and $f : E \to E_1$ a map. Using theorem 7.2 and the map f, we get a $G(E_1)$-valued model of the theory $Sp(E)$. Hence there is a unique σ-map from B_0 to $G(E_1)$. By theorem 2.7, there is a unique corresponding σ-map from $F(B_0)$ to E_1. □

We conclude this section by an interpretation of Radon-Nikodym's theorem in this framework. Let B_0 be the σ-complete Boolean algebra of components of the space M of I-bounded measure, as in theorem 4.1. We can see B_0 as the measure algebra of I: the Boolean algebra of measurable subsets quotiented by sets of measure 0. If $b \in B_0$ the measure of b is $\psi(b)$. A *bounded measurable function* can now be defined to be an element of $F(B_0)$. By theorem 7.2 we have an isomorphism between M and $F(B_0)$. Hence any element of M (I-bounded measure) can be seen as a measurable function (the derivative of this measure).

Future Work

We have given a map $\phi_1 : V_1 \to M$ from bounded Baire functions to bounded measurable functions. One can ask if this map is surjective. This amounts to a refinement of Radon-Nikodym's theorem, where we prove that the derivative of a measure can be taken to be a bounded Baire function. It seems possible to get this refinement if we choose for R the set of hyperarithmetical reals, i.e. the free Dedekind σ-complete Riesz space. In this case any R-valued step function on B is represented by an element of V_1, because V_1 can be seen as a R-vector space. It should then be possible to represent an element $l \in [0, I]$ as a limsup of R-valued step functions: if B is seen as a direct limit of finite Boolean algebras, we take for step functions the functions

$$b \longmapsto l(b)/I(b)$$

which approximate the derivative of l. Classically, this would be a Baire function of level 2. Here this is not the case a priori because the step functions are R-valued. The proof of this would follow the classical proof, which shows that the Borel set of points for which this sequence converges is of measure 1.

Another direction of research will be to relate our work to measure in ordered vector spaces. We think that our work gives an alternative way of obtaining the results of [18, 19]. We only point out the formulation of the example in [18] showing that one cannot expect the measure to be regular in this framework: let B_1 be the algebra of Borel sets of Ω and N the σ-ideal of meager sets.[5] We have a measure $s : B_1 \to B_1/N$ which takes values in $E = F(B_1/N)$. If X is the set of all periodic sequences, then $s(X) = 0$ because X is countable, while if $U \supseteq X$ is open we have $\bar{U} = \Omega$ because X is dense and hence $s(U) = 1$. Thus s is an E-valued measure which is not regular.

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