Real Spectrum

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Abstract

To any commutative ring, we associate an entailment relation which describes its real spectrum [Joh82].

Introduction

Johnstone gives an axiomatic description of the real spectrum of a commutative ring [Joh82]: this formal space is described as a propositional theory. In this note we describe an entailment relation which gives a complete characterisation of this theory.

1 Preliminary

Let A be a commutative ring. A *cone* of A is a subset $C \subseteq A$ closed by addition, mutiplication and which contains all square elements $x^2,\ x \in A$. The following claims are directly checked: the smallest cone S of A is the set of sum of squares; if C is a cone and $a \in A$ the cone generated by C and a, that is the least cone containing C and a is the set $C + aC$ of elements $u + va,\ u, v \in C$.

The multiplicative monoid generated by elements $x_1, \ldots, x_n$ is the set of all elements of the form $x_1^{k_1} \ldots x_n^{k_n}$ with non negative integers $k_1, \ldots, k_n$.

2 Entailment Relation

We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ arbitrary finite sets of A and we define the following relation on these finite sets: $\{x_1, \ldots, x_n\} \vdash \{y_1, \ldots, y_m\}$ iff there exists m in the monoid generated by $x_1, \ldots, x_n$ and p in the cone generated by $x_1, \ldots, x_n, -y_1, \ldots, -y_m$ such that $m + p = 0$.

We write $\mathcal{A}, \mathcal{B}$ for $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}, x$ for $\mathcal{A} \cup \{x\}$.

Theorem 2.1 *The relation $\vdash$ satisfies the following conditions of reflexivity, monotonicity and transitivity*

$$\mathcal{A} \vdash \mathcal{B}\ if\ \mathcal{A} \cap \mathcal{B} \neq \emptyset \tag{R}$$

$$\frac{\mathcal{A} \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A}' \vdash \mathcal{B}, \mathcal{B}'} \tag{M}$$

$$\frac{\mathcal{A} \vdash y, \mathcal{C} \quad \mathcal{A}, y \vdash \mathcal{C}}{\mathcal{A} \vdash \mathcal{C}} \tag{T}$$

Thus the relation $\vdash$ is an entailment relation [Sco74].

Proof. Only the rule (T) is not direct. We show that if M is a multiplicative monoid and C a cone such that $M \subseteq C$ and $x \in A$ is such that we have some relations

$$m_1 + u_1 + (-x)v_1 = 0 \qquad\qquad m_2 x^k + u_2 + xv_2 = 0$$

with $m_1, m_2 \in M$ and $u_1, v_1, u_2, v_2 \in P$ then there is a relation $m+u = 0$ with $m \in M,\ u \in C$. For this let $M' = M + C$ be the set of all elements of the form $m + u,\ m \in M,\ u \in C$. Since $M \subseteq C$ the set M' is closed by multiplication. We can rewrite the first relation as $m'_1 = xv_1$ with $m'_1 = m_1 + u_1 \in M'$. The second relation implies then $m_2(xv_1)^k + u_2 v_1^k + xv_2 v_1^k = 0$ and hence is of the form $m'_2 + xv = 0$ for some $m'_2 \in M'$ and $v \in C$. It follows that $m'_1 m'_2 + x^2 v_1 v = 0$ which is of the form $m' = 0$ for some $m' = m'_1 m'_2 + x^2 v_1 v \in M'$ as desired. □

Intuitively the relation $x_1, \ldots, x_n \vdash y_1, \ldots, y_m$ can be read as

$$x_1 > 0 \wedge \ldots \wedge x_n > 0 \rightarrow y_1 > 0 \vee \ldots \vee y_m > 0$$

where $>$ is a generic strict total ordering on the ring A. This is expressed formally by the following result.

Theorem 2.2 *The relation $\vdash$ is the least entailment relation such that*

1. $\vdash 1$,
2. $x, -x \vdash$,
3. $x + y \vdash x, y$,
4. $x, y \vdash xy$,
5. $xy \vdash x, -x$,
6. $xy \vdash x, -y$.

References

[Joh82] P. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[Sco74] D. Scott. Completeness and axiomatiszability. *Proceedings of the Tarski Symposium*, pages 411–435, 1974.