Presheaf model of type theory

The set theoretic model of type theory interprets universes à la Russell. The (pre)sheaf models do not validate these universes. However we can validate a simpler version than universes à la Tarski, and this is what we present here, in the case of presheaf models.

1 Syntax

We list the rules of type theory, using a name-free syntax.

$$\frac{\Gamma \vdash}{1 : \Gamma \to \Gamma} \qquad \frac{\sigma : \Delta \to \Gamma \quad \delta : \Theta \to \Delta}{\sigma\delta : \Theta \to \Gamma}$$

$$\frac{\Gamma \vdash A \ \mathsf{type}_n \quad \sigma : \Delta \to \Gamma}{\Delta \vdash A\sigma \ \mathsf{type}_n} \qquad \frac{\Gamma \vdash t : A \quad \sigma : \Delta \to \Gamma}{\Delta \vdash t\sigma : A\sigma}$$

$$\frac{}{() \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A \ \mathsf{type}_n}{\Gamma.A \vdash} \qquad \frac{\Gamma \vdash A \ \mathsf{type}_n}{\mathsf{p} : \Gamma.A \to \Gamma} \qquad \frac{\Gamma \vdash A \ \mathsf{type}_n}{\Gamma.A \vdash \mathsf{q} : A\mathsf{p}}$$

$$\frac{\sigma : \Delta \to \Gamma \quad \Gamma \vdash A \quad \Delta \vdash u : A\sigma}{(\sigma, u) : \Delta \to \Gamma.A}$$

$$\frac{\Gamma \vdash A \ \mathsf{type}_n \quad \Gamma.A \vdash B \ \mathsf{type}_n}{\Gamma \vdash \ \Pi \ A \ B \ \mathsf{type}_n} \qquad \frac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda b : \Pi \ A \ B} \qquad \frac{\Gamma \vdash w : \Pi \ A \ B \quad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}(w, u) : B[u]}$$

$$\frac{\Gamma \vdash A \ \mathsf{type}_n \quad \Gamma.A \vdash B \ \mathsf{type}_n}{\Gamma \vdash \ \Sigma \ A \ B \ \mathsf{type}_n} \qquad \frac{\Gamma \vdash u : A \quad \Gamma \vdash v : B[u]}{\Gamma \vdash (u, v) : \Sigma \ A \ B} \qquad \frac{\Gamma \vdash w : \Sigma \ A \ B}{\Gamma \vdash \mathsf{p}w : A} \qquad \frac{\Gamma \vdash w : \Sigma \ A \ B}{\Gamma \vdash \mathsf{q} \ w : B[\mathsf{p}w]}$$

where $[u] = (1, u) : \Gamma \to \Gamma.A$ if $\Gamma \vdash u : A$.

$$1\sigma = \sigma = \sigma 1 \qquad (\sigma\delta)\nu = \sigma(\delta\nu)$$

$$(\sigma, u)\delta = (\sigma\delta, u\delta) \qquad \mathsf{p}(\sigma, u) = \sigma \qquad \mathsf{q}(\sigma, u) = u$$

$$(A\sigma)\delta = A(\sigma\delta) \qquad A1 = A \qquad (a\sigma)\delta = a(\sigma\delta) \qquad a1 = a$$

$$\mathsf{app}(w, u)\delta = \mathsf{app}(w\delta, u\delta) \qquad \mathsf{app}(\lambda b, u) = b[u] \qquad (\lambda b)\sigma = \lambda(b(\sigma\mathsf{p}, \mathsf{q}))$$

$$u, v)\delta = (u\delta, v\delta) \qquad \mathsf{p}(u, v) = u \qquad \mathsf{q}(u, v) = v \qquad (\mathsf{p}u)\sigma = \mathsf{p}(u\sigma) \qquad (\mathsf{q}u)\sigma = \mathsf{q}(u\sigma)$$

$$1 = (\mathsf{p}, \mathsf{q}) \qquad v = \lambda \mathsf{app}(v\mathsf{p}, \mathsf{q})$$

We add the following rules for universes.

$$\frac{\Gamma \vdash A \ \mathsf{type}_n}{\Gamma \vdash |A| : U_n} \qquad \frac{\Gamma \vdash T : U_n}{\Gamma \vdash El \ T \ \mathsf{type}_n}$$

$$\frac{\Gamma \vdash A \ \mathsf{type}_n}{\Gamma \vdash A \ \mathsf{type}_{n+1}} \qquad \frac{\Gamma \vdash T : U_n}{\Gamma \vdash T : U_{n+1}}$$

$$\frac{}{\Gamma \vdash U_n \ \mathsf{type}_{n+1}}$$

$$El \ |A| = \ A \qquad |El \ T| = T$$

With this presentation, we can define $\pi \ T \ V = |\Pi \ (El \ T) \ (El \ V)|$ if $\Gamma \vdash T : U_n$ and $\Gamma.El \ T \vdash V : U_n$. This satisfies $El \ (\pi \ T \ V) = \Pi \ (El \ T) \ (El \ V)$.

2 Presheaf model

If $\mathcal{C}$ is any small category, the presheaf model of type theory over $\mathcal{C}$ can be described as follows.

To simplify the presentation, we don't consider the question of size.

We write $X, Y, Z, \ldots$ the objects of $\mathcal{C}$ and $f, g, h, \ldots$ the maps of $\mathcal{C}$. If $f : X \to Y$ and $g : Y \to Z$ we write gf the composition of f and g. We write $1_X : X \to X$ or simply $1 : X \to X$ the identity map of X. Thus we have $(fg)h = f(gh)$ and $1f = f1 = f$.

A context is interpreted by a presheaf Γ: for any object X of $\mathcal{C}$ we have a set $\Gamma(X)$ and if $f : Y \to X$ we have a map $\rho \longmapsto \rho f,\ \Gamma(X) \to \Gamma(Y)$. This should satisfy $\rho 1 = \rho$ and $(\rho f)g = \rho(fg)$ for $f : Y \to X$ and $g : Z \to Y$.

A type $\Gamma \vdash A$ over Γ is given by a set $A\rho$ for each $\rho : \Gamma(X)$. Furthermore if $f : Y \to X$ we have $\rho f : \Gamma(Y)$ and we can consider the set $A\rho f$. We should have a map $u \longmapsto uf,\ A\rho \to A\rho f$ which should satisfy $u1 = u$ and $(uf)g = u(fg)$.

An element $\Gamma \vdash a : A$ is interpreted by a family $a\rho : A\rho$ such that $(a\rho)f = a(\rho f)$ for any $\rho : \Gamma(X)$ and $f : Y \to X$.

This can be seen as a concrete description of what is respectively a fibration and a section of this fibration.

If $\Gamma \vdash A$ we can define a new presheaf $\Gamma.A$ by taking $(\rho, u) : (\Gamma.A)(X)$ to mean $\rho : \Gamma(X)$ and $u : A\rho$. We define $(\rho, u)f = \rho f, uf$.

If we have a map $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ we define $\Delta \vdash A\sigma$ by $(A\sigma)\rho = A(\sigma\rho)$.

If $\Gamma \vdash A$ and $\rho : \Gamma(X)$ we define $|A|\rho$ to be the family $(A\rho f,\ f : Y \to X)$ with restriction map $A\rho f \to A\rho f g,\ u \longmapsto ug$ for $g : Z \to Y$.

We define $U(X)$ as the set of families of sets $Pf,\ f : Y \to X$ together with restriction maps $Pf \to Pfg,\ u \longmapsto ug$ satisfying $u1 = u$ and $(ug)h = u(gh)$. We define then $\Gamma \vdash U$ by taking $U\rho = U(X)$ if $\rho : \Gamma(X)$.

If we have $\Gamma \vdash T : U$ we define $\Gamma \vdash El\ T$ by the equation $(El\ T)\rho = T\rho 1_X$ for $\rho : \Gamma(X)$.

We validate then $|El\ T| = T$ and $El\ |A| = A$.

If $\Gamma \vdash A$ we have $(El\ |A|)\rho = |A|\rho 1_X$ and $|A|\rho$ is the family $A\rho f,\ f : \to X$, so that $|A|\rho 1_X = A\rho 1_X = A\rho$. The restriction map $u \longmapsto uf,\ (El\ |A|)\rho \to (El\ |A|)\rho f$ is the restriction map defined by $A\rho \to A\rho f$.

If $\Gamma \vdash T : U$ the family $(El\ T)\rho f,\ f : Y \to X$ is defined by $T(\rho f)1_Y = T\rho f$, and so $|El\ T| = T$.

We can interpret dependent products $\Gamma \vdash \Pi\ A\ B$ and sums $\Gamma \vdash \Sigma\ A\ B$ if we have $\Gamma \vdash A$ and $\Gamma.A \vdash B$. For $\rho : \Gamma(X)$ we define $(u, v) : (\Sigma\ A\ B)\rho$ to mean $u : A\rho$ and $v : B(\rho, u)$. We define $(u, v)f = uf, vf$ for $f : Y \to X$. On the other hand an element of $(\Pi\ A\ B)\rho$ is a family w indexed by $h : Y \to X$ with

$$wh : \prod_{u : A\rho h} B(\rho h, u)$$

and such that $\mathsf{app}(wh, u)g = \mathsf{app}(whg, ug)$ if $h : Y \to X$ and $g : Z \to Y$. We define then $(wh)f = w(hf)$. We write $w = w1$.

We can interpret $\Gamma \vdash \lambda t : \Pi\ A\ B$ whenever $\Gamma.A \vdash t : B$ and $\Gamma \vdash \mathsf{app}(v, u) : B[u]$ if $\Gamma \vdash u : A$ and $\Gamma \vdash v : \Pi\ A\ B$. Here we write $[u]$ the map $\Gamma \to \Gamma.A$ defined by $[u]\rho = \rho, u\rho$. If $\rho : \Gamma(X)$ and $f : Y \to X$ we define $\mathsf{app}((\lambda t)\rho f, a) = t(\rho f, a) : B(\rho f, a)$ for $a : A\rho f$. We take $\mathsf{app}(v, u)\rho = \mathsf{app}(v\rho, u\rho) : B(\rho, u\rho)$. We can then check that we have

$$\mathsf{app}(\lambda t, u)\rho = t(\rho, u\rho) = t[u]\rho : B(\rho, u\rho)$$

if $\Gamma.A \vdash t : B$ and $\Gamma \vdash u : A$ and $\rho : \Gamma(X)$, which shows that the model validates the conversion rule $\Gamma \vdash \mathsf{app}(\lambda t, u) = t[u] : B[u]$.