Valuations and Dedekind's Prague Theorem

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Abstract

To any field K we associate an *entailment relation* in the sense of Scott [12]. In this way we can interpret an abstract propositional theory representing a generic valuation ring of a field, and obtain a simple effective proof of Dedekind's Prague theorem [5,6].

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1 Introduction

To any field K we associate a relation $\vdash$ between finite sets of non zero elements of K which satisfy the three conditions of an *entailment relation* in the sense of Scott [12], and some further simple conditions. In this way, we can give constructive sense of a generic valuation ring of a field. Alternatively, this can be seen as a generalisation of the notion of integral element, and this notion can be used to prove that a given element is integral. As an example, we present a simple effective proof of Dedekind's Prague theorem.

2 Valuations

Let K be a field, that is a commutative ring in which any element is 0 or is invertible. We write S the set of invertible elements of K. We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ arbitrary finite subsets of S and we define the following relation on these finite sets:

$$\{y_1, \ldots, y_m\} \vdash \{x_1, \ldots, x_n\}$$

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iff there exist $q_1, \dots, q_n$ polynomials in $\mathbb{Z}[x_1^{-1}, \dots, x_n^{-1}, y_1, \dots, y_m]$ such that

$$x_1^{-1}q_1 + \dots + x_n^{-1}q_n = 1.$$

We write $\mathcal{A}, \mathcal{B}$ for $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}, x$ for $\mathcal{A} \cup \{x\}$.

Theorem 1 *The relation $\vdash$ satisfies the following conditions of reflexivity, monotonicity and transitivity*

$$\mathcal{A} \vdash \mathcal{B} \text{ } if \text{ } \mathcal{A} \cap \mathcal{B} \neq \emptyset \tag{R}$$

$$\frac{\mathcal{A} \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A}' \vdash \mathcal{B}, \mathcal{B}'} \tag{M}$$

$$\frac{\mathcal{A} \vdash y, \mathcal{C} \qquad \mathcal{A}, y \vdash \mathcal{C}}{\mathcal{A} \vdash \mathcal{C}} \tag{T}$$

Thus the relation $\vdash$ is an entailment relation [12].

Proof. The condition (R) and (M) are clear from the definition. The condition (T) can be seen as an elimination of the variable y, and follows from the Sylvester resultant method. Indeed, to have $y_1, \dots, y_m \vdash x_1, \dots, x_n$ can be restated as the existence of a relation

$$x_1^{k_1} \dots x_n^{k_n} = P(x_1, \dots, x_n)$$

where P is a polynomial in $\mathbb{Z}[y_1, \dots, y_m][x_1, \dots, x_n]$ of degree $< (k_1, \dots, k_n)$ (for the product ordering). If we have both $y_1, \dots, y_m \vdash x_1, \dots, x_n, y$ and $y, y_1, \dots, y_m \vdash x_1, \dots, x_n$ this implies the existence of two relations:

$$(x_1^{k_1} \dots x_n^{k_n} + p_l)y^l + p_{l-1}y^{l-1} + \dots + p_0 = 0$$

$$q_r y^r + \dots + (x_1^{t_1} \dots x_n^{t_n} + q_0) = 0$$

where the degree of $p_{l-1}, \dots, p_0$ are all $\leq (k_1, \dots, k_n)$, the degree of p_l is $< (k_1, \dots, k_n)$ and the degree of $q_r, \dots, q_0$ are all $< (t_1, \dots, t_n)$. It follows that the resultant of these two polynomials in y will give a witness for $y_1, \dots, y_m \vdash x_1, \dots, x_n$.

In the examples we shall analyse below, we always have $l = r = 1$. $\square$

Proposition 2 *The relation $\vdash$ satisfies the following conditions*

$$\vdash x, y \qquad if \text{ } xy = x + y \tag{1}$$

$$\vdash x, x^{-1} \tag{2}$$

$$x \vdash -x \tag{3}$$

$$x, y \vdash xy \tag{4}$$

These conditions are immediately verified: for instance the first condition follows from $x^{-1}+y^{-1}=1$. The importance of this proposition is shown by the following observation, which has also a direct proof.

Proposition 3 *The entailment relation* $\vdash$ *is the least entailment relation on* S *satisfying the conditions* $(1),(2),(3)$ *and* (4).

Proof. Let $\vdash$ be an arbitrary entailment relation satisfying the conditions $(1),(2),(3)$ and (4). We first show $xy \vdash x, y$. This follows by $(T),(M)$ from $xy, y^{-1} \vdash x$ and $\vdash y, y^{-1}$, which follows from (4) and (1) respectively.

By induction, this implies $u_1 \dots u_n \vdash u_1, \dots, u_n$.

Next we show that we have $\vdash u_1^{-1}, \dots, u_n^{-1}$ if $u_1 + \dots + u_n = 1$. This is the condition (1) if $n = 2$. The induction step follows from $(x+y)^{-1} \vdash x^{-1}, y^{-1}$ which is a consequence of $(T),(M)$ and the derivations

- $\vdash (x+y)x^{-1}, (x+y)y^{-1}$ by (1),
- $(x+y)^{-1}, (x+y)x^{-1} \vdash x^{-1}$ by (4), and
- $(x+y)^{-1}, (x+y)y^{-1} \vdash y^{-1}$ by (4).

Suppose now we have

$$1 = x_1^{-1}p_1 + \dots + x_n^{-1}p_n$$

for some polynomials $p_1, \dots, p_n$ in x_i^{-1}, y_j. It would follow that we have a relation of the form $\vdash \dots, u_k v_k^{-1}, \dots$ where the u_k are product of some x_i and v_k product of some y_j. It follows from (4) that we have $y_1, \dots, y_m \vdash \dots, u_k, \dots$ and hence $y_1, \dots, y_m \vdash x_1, \dots, x_n$ since, by above, for each k we have $u_k \vdash x_1, \dots, x_n$. $\square$

The theorem can be used to give a constructive interpretation of the space of valuations of a field. Actually, we have the following suggestive interpretation of the relation $\vdash$. We recall that a *valuation ring* A of K is a subring of K such that, for any element x of S we have $x \in A$ or $x^{-1} \in A$. We can now interpret $y_1, \dots, y_m \vdash x_1, \dots, x_n$ as

$$\text{if } y_1 \in A, \text{ and } \dots, \text{ and } y_m \in A, \text{ then } x_1 \in A, \text{ or } \dots, \text{ or } x_n \in A,$$

where A is a "generic" valuation ring of K. Classically, these two assertions are equivalent. We can see that, in our approach, valuation rings appear only as a system of notations for recording the existence of polynomial identities.

We will also write $x \leq y$ instead of yx^{-1}. This notation is justified since $yx^{-1} \in A$ defines an ordering relation if A is a valuation ring.

Lemma 4 *We have* $\vdash x \leq x$, *and* $x \leq y, y \leq z \vdash x \leq z$, *and* $\vdash x \leq y, y \leq x$.

This is direct from the proposition 2. This can be interpreted as saying that $\leq$ defines a total ordering on S.

Let now be $u_1, \ldots, u_n, w$ be any elements of S :

Lemma 5 *If* $w = u_1 + \cdots + u_n$ *then* $\vdash u_1 \leq w, \ldots, u_n \leq w$.

Proof. This is witnessed by the identity $1 = u_1 w^{-1} + \cdots + u_n w^{-1}$. □

In term of valuation rings, this expresses that we have

$$wu_1^{-1} \in A, \ or \ \ldots, \ or \ wu_n^{-1} \in A,$$

whenever $w = u_1 + \cdots + u_n$.

3 Applications

Notice that $y_1, \ldots, y_m \vdash x$ iff there is a relation $x^k = q_{k-1}x^{k-1} + \cdots + q_0$ where q_j are polynomials in $y_1, \ldots, y_m$ with integer coefficients. In other terms, $y_1, \ldots, y_m \vdash x$ iff x is integral over $\mathbb{Z}[y_1, \ldots, y_m]$. We can thus use the results above as a method to prove that an element is integral over the subring generated by other elements. Here is a typical example, where we work with $K = \mathbb{Q}(a_0, \ldots, a_m, b_0, \ldots, b_n)$ where a_i and b_j are indeterminates. The following is a theorem of Kronecker.

Theorem 6 *If* $c_k = \sum_{i+j=k} a_i b_j$ *then each element* $a_i b_j$ *is integral over the subring generated by* $c_0, \ldots, c_{m+n}$.

Proof. Non-constructively, this can be reduced to the fact that $a_i b_j$ belongs to any valuation ring A of K that contains $c_0, \ldots, c_{m+n}$. This in turn can be proved following the proof of Gauss' lemma [10]. Since $\leq$ is a total ordering, we can find i_0 and j_0 such that $a_{i_0} \leq a_i$ for all i and $b_{j_0} \leq b_j$ for all j (we recall that $x \leq y$ means $yx^{-1} \in A$). Furthermore, we can suppose that i_0 is the greatest number with the property that a_{i_0} is minimum among a_i (and similarly for j_0.)

It is then enough to show that $a_{i_0} b_{j_0} \in A$, since we have $a_i b_j (a_{i_0} b_{j_0})^{-1} \in A$ for all i, j. For this we use the identity

$$a_{i_0} b_{j_0} = c_{k_0} - \sum_{i+j=k_0,\ i_0<i} a_i b_j - \sum_{i+j=k_0,\ j_0<j} a_i b_j$$

where $k_0 = i_0 + j_0$. It follows from this identity and the lemma 5 that we have

- $c_{k_0} \leq a_{i_0} b_{j_0}$ or
- $a_i b_j \leq a_{i_0} b_{j_0}$ for some $i > i_0$ or
- $a_i b_j \leq a_{i_0} b_{j_0}$ for some $j > j_0$.

By the choice of i_0, j_0 the two last cases are not possible. Hence, we are in the first case, and we have $a_i b_j \in A$ for all i, j. □

The non-constructive use of valuation ring does not allow us however to extract a polynomial identity satisfied by each $a_i b_j$. But the entailment relation $\vdash$ gives exactly what is needed in order to make a constructive reinterpretation of this reasoning. It shows us how to read this argument as a proof of the relation $c_0, \ldots, c_{m+n} \vdash a_i b_j$, which in turn expresses the existence of a polynomial identity showing that the element $a_i b_j$ is integral over the subring generated by $c_0, \ldots, c_{m+n}$.

Let us illustrate this reasoning. We will write $\mathcal{C}$ for $c_0, \ldots, c_{m+n}$ and show in two cases how to derive $\mathcal{C} \vdash a_0 b_1$.

First in the case $n = m = 1$. The non-constructive argument is in this case the following. Let A be a valuation ring containing c_0, c_1, c_2. We have $a_0 a_1^{-1} \in A$ or $a_1 a_0^{-1} \in A$. In the first case $a_0 b_1 = c_2 a_0 a_1^{-1} \in A$ and in the second case $a_0 b_1 = c_1 - c_0 a_1 a_0^{-1} \in A$. So in both cases $a_0 b_1 \in A$. If we read the same argument, using the entailment relation $\vdash$ instead of valuation rings, this argument becomes:

- $\vdash a_0 \leq a_1, a_1 \leq a_0$,
- $\mathcal{C}, a_0 \leq a_1 \vdash a_0 b_1$ and
- $\mathcal{C}, a_1 \leq a_0 \vdash a_0 b_1$.

Hence $\mathcal{C} \vdash a_0 b_1$ follows by the rule (T).

Each step records the existence of a polynomial identity, that can readily be made explicit. Namely $\vdash a_0 \leq a_1, a_1 \leq a_0$ is witnessed by the identity $1 = xx^{-1}$ where $x = a_1 a_0^{-1}$. On the other hand $\mathcal{C}, a_0 \leq a_1 \vdash a_0 b_1$ is witnessed by the identity $a_0 b_1 = c_1 - x c_0$ while $\mathcal{C}, a_1 \leq a_0 \vdash a_0 b_1$ is witnessed by the identity $a_0 b_1 = x^{-1} c_2$. If we eliminate x between these two last identities we get $a_0 b_1 = c_1 - c_2 (a_0 b_1)^{-1} c_0$ which shows explicitly that $a_0 b_1$ is integral over the ring generated by c_0, c_1, c_2.

In the case $n = m = 2$, we obtain the following derivation of $\mathcal{C} \vdash a_0 b_1$. First, we have $\mathcal{C}, b_0 \leq b_1 \vdash a_0 b_1$ because $b_1 a_0 = c_0 b_1 b_0^{-1}$. Hence, since we have $\vdash b_0 \leq b_1, b_1 \leq b_0$ it is enough by (T) to show $\mathcal{C}, b_1 \leq b_0 \vdash b_1 a_0$. By a similar reasoning, it is enough to show $\mathcal{C}, b_1 \leq b_0, b_2 \leq b_1 \vdash b_1 a_0$ and

$\mathcal{C}, b_1 \leq b_0, b_1 \leq b_2 \vdash b_1 a_0$.

Here is a derivation of $\mathcal{C}' \vdash b_1 a_0$, where $\mathcal{C}' = \mathcal{C}, b_1 \leq b_0, b_2 \leq b_1$:

(i) $\mathcal{C}', a_1 \leq a_0 \vdash b_1 a_0$, by $b_1 a_0 = b_1 b_2^{-1} a_0 a_1^{-1}(c_3 - c_4 b_1 b_2^{-1})$,
(ii) $\mathcal{C}', a_2 \leq a_0 \vdash b_1 a_0$, by $b_1 a_0 = c_4 a_0 a_2^{-1} b_1 b_2^{-1}$,
(iii) $\mathcal{C}' \vdash b_1 a_0, a_1 \leq a_0, a_2 \leq a_0$, by
$b_1 a_0 = b_1 b_2^{-1}(c_2 - b_1 a_0 a_1 a_0^{-1} - b_1 a_0 b_0 b_1^{-1} a_2 a_0^{-1})$,
(iv) $\mathcal{C}' \vdash b_1 a_0$, by the three above and (T).

The identity for the step (iv) can be found by eliminating $a_0 a_1^{-1}$ and $a_0 a_2^{-1}$ between the identities for the steps (ii) and (iii). In this way we get the identity

$$b_1 a_0 = b_1 b_2^{-1}(c_2 - b_1 b_2^{-1}(c_3 - c_4 b_1 b_2^{-1}) - c_4 b_0 b_1^{-1} b_1 b_2^{-1}).$$

Here is a derivation of $\mathcal{C}'' \vdash b_1 a_0$, where $\mathcal{C}'' = \mathcal{C}, b_1 \leq b_0, b_1 \leq b_2$:

(i) $\mathcal{C}'' \vdash b_1 a_0, a_1 \leq a_0$, by $b_1 a_0 = c_1 - b_1 a_0 b_0 b_1^{-1} a_1 a_0^{-1}$,
(ii) $\mathcal{C}'', a_1 \leq a_0 \vdash b_1 a_0, a_2 \leq a_1$, by
$b_1 a_0 = a_0 a_1^{-1}(c_2 - b_1 a_1 b_2 b_1^{-1} a_0 a_1^{-1} - b_1 a_1 b_0 b_1^{-1} a_2 a_1^{-1})$,
(iii) $\mathcal{C}'', a_1 \leq a_0, a_2 \leq a_1 \vdash b_1 a_0$, by $b_1 a_0 = a_1 a_2^{-1} a_0 a_1^{-1}(c_3 - b_1 a_2 b_2 b_1^{-1} a_1 a_2^{-1})$,
(iv) $\mathcal{C}'', a_1 \leq a_0 \vdash b_1 a_0$, by (ii),(iii) and (T),
(v) $\mathcal{C}'' \vdash b_1 a_0$, by (i),(iv) and (T),

Altogether, this gives the identity

$$(a_0 b_1)^6 = p_1 (a_0 b_1)^5 + p_2 (a_0 b_1)^4 + p_3 (a_0 b_1)^3 + p_4 (a_0 b_1)^2 + p_5 (a_0 b_1) + p_6$$

where

$$p_1 = 3c_1, \quad p_2 = -3c_1^2 - 2c_0 c_2, \quad p_3 = c_1^3 + 4c_0 c_1 c_2,$$

$$p_4 = -c_0^2 c_1 c_3 - 2c_0 c_1^2 c_2 - c_0^2 c_2^2 + 4c_0^3 c_4,$$

$$p_5 = c_0^2 c_1^2 c_3 + c_0^2 c_1 c_2^2 - 4c_0^3 c_1 c_4, \quad p_6 = -c_0^3 c_1 c_2 c_3 + c_0^4 c_3^2 + c_0^3 c_1^2 c_4,$$

which is the same as the one given in [6], and this computation can be done without computers. We find it interesting that the computations underlying this argument, a priori not effective, are actually quite feasible in this case.

Notice that this identity is "homogeneous" in the sense that the coefficient p_n is a linear combination of monomials $c_0^{i_0} \dots c_4^{i_4}$ with $i_0 + \cdots i_4 = n$. That such an "homogeneous" identity exists follows by replacing a_i by $a_i X$ and c_k by $c_k X$ where X is a new indeterminate. Similarly, by replacing a_i by $a_i X Y^i$, and b_j by $b_j Y^j$ and c_k by $c_k X Y^k$ one can show the existence of an identity where each p_n is a linear combination of monomials $\prod_k c_k^{i_k}$ with $\sum_k i_k = n$ *and* $\sum_k k i_k = n$.

We can look at the computation required in other proofs of this theorem, due to Dedekind [11], or Hurwitz [6]; they involve computation of a 6 by 6 determinant. For instance, Hurwitz's argument in this case uses the following identities:

$$\begin{array}{rlllllll}
(a_0b_1)M_1 = & c_1M_1 & -c_0M_2 & & & & & \\
(a_0b_1)M_2 = & & c_1M_2 & & -c_0M_4 & & & \\
(a_0b_1)M_3 = & & & c_1M_3 & & -c_0M_5 & & \\
(a_0b_1)M_4 = & -c_3M_1 & +c_2M_2 & +c_1M_3 & & -2c_0M_5 & & \\
(a_0b_1)M_5 = & -c_4M_1 & & +c_2M_3 & & & -c_0M_6 & \\
(a_0b_1)M_6 = & & -c_4M_2 & +c_3M_3 & & & &
\end{array}$$

where

$$M_1 = a_0^2,\ M_2 = a_0a_1,\ M_3 = a_0a_2,\ M_4 = a_1^2,\ M_5 = a_1a_2,\ M_6 = a_2^2.$$

This theorem implies directly Dedekind's Prague theorem [5,6].

Theorem 7 *Let f and g be polynomials in one indeterminate whose coefficients are algebraic numbers. If all coefficients of fg are algebraic integers, then the product of any coefficient of f with any coefficient of g is an algebraic integer.*

Another corollary is the following result, used at the very beginning of Hilbert's "ZahlBericht".

Theorem 8 *Let f and g be polynomials in one indeterminate whose coefficients are algebraic numbers. If all coefficients of fg are divisible by some algebraic integer, integers, then the product of any coefficient of f with any coefficient of g is divisible by this algebraic number.*

4 Related Work

4.1 Theory of Locale

We can present in the framework of entailment relations another example of a space associated to an algebraic structure, namely the *spectrum* of an arbitrary commutative ring R. It will be the entailment relation on R such that $y_1, \ldots, y_m \vdash x_1, \ldots, x_n$ iff the product $y_1 \ldots y_m$ is in the product of the radical ideal generated by $x_1, \ldots, x_n$. Intuitively $\vdash x$ means $x \in P$ where P is the complement of a generic prime ideal on R. This theory has been analysed from the point of view of the theory of locales: see [7–9] and [1]. We indicate now some connections with the present work.

First, any entailment relation defines canonically a distributive lattice and hence a spectral space (see [2] for the details). Let R be an integral ring, and K its field of fraction, and as before S the set of invertible elements of K. We can associate to R two entailment relations, each of them defining a distributive lattice:

- its spectrum $Sp(R)$,
- the *abstract Riemann surface* $X(R)$ [4] which can be defined as the entailment relation on S such that $y_1, \ldots, y_m \vdash x_1, \ldots, x_n$ iff there exists $q_1, \ldots, q_n$ polynomials *in* $R[x_i^{-1}, y_j]$ such that $x_1^{-1}q_1 + \cdots + x_n^{-1}q_n = 1$.

Next, following [4] we can connect the two lattices by defining a morphism $\phi : Sp(R) \to X(R)$ taking $\phi(0) = 0$ and $\phi(r) = r^{-1}$ if $r \in R,\ r \neq 0$. This corresponds to the continuous map $f_R : X(R) \to Sp(R)$ defined in [4], which to any valuation ring containing R associates its center on R. Notice that the surjectivity of the map f_R gets an elementary interpretation as the fact that the map ϕ satisfies $\phi(x) \leq \phi(y)$ iff $x \leq y$.

Proposition 9 *For any non-zero elements x_i, y_j in R we have $y_1, \ldots, y_m \vdash x_1, \ldots, x_n$ in $Sp(R)$ iff $\phi(y_1), \ldots, \phi(y_m) \vdash \phi(x_1), \ldots, \phi(x_n)$ in $X(R)$.*

Proof. Direct from the definitions. □

One can then use such an interpretation to reduce theorem 6 to the localic formulation of Gauss' lemma presented in [9] and [1]: taking $R = \mathbb{Z}[a_i/a_{i_0}, b_j/b_{j_0}]$, the localic version of Gauss' lemma shows that we have

$$\bigvee_k c_k/a_{i_0}b_{j_0} = (\bigvee_i a_i/a_{i_0})(\bigvee_j b_j/b_{j_0})$$

in the lattice $Sp(R)$ and hence by the interpretation ϕ we have in $X(R)$

$$\vdash \bigvee_k a_{i_0}b_{j_0}/c_k$$

which implies that in the space of valuations of K

$$\bigwedge_i a_i/a_{i_0}, \bigwedge_j b_j/b_{j_0} \vdash \bigvee_k a_{i_0}b_{j_0}/c_k$$

and hence

$$\bigwedge_i a_i/a_{i_0}, \bigwedge_j b_j/b_{j_0}, \bigwedge_k c_k \vdash a_{i0}b_{j0}$$

Since this holds for any i_0, j_0 we deduce

$$\bigwedge_k c_k \vdash a_i b_j$$

as desired.

4.2 Dynamical Algebras

Our work was inspired by the work of Coste, Lombardi and Roy on dynamical methods in algebra [3]. The proposition $x \in A$ for a generic valuation ring corresponds to their proposition $Vr(x)$. The definition of the entailment relation suggests the following generalisation of their corollary 4.16: if $Vr(v_1) \wedge \cdots \wedge Vr(v_m) \vdash Vr(u_1) \vee \cdots \vee Vr(u_n)$ is provable in the theory of valued fields then we can find polynomials $p_1, \dots, p_n$ in the variables $v_1, \dots, v_m, u_1^{-1}, \dots, u_n^{-1}$ with integer coefficients such that $u_1^{-1}p_1 + \cdots + u_n^{-1}p_n = 1$.

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References

[1] B. Banaschewski and J. J. C. Vermeulen. Polynomials and radical ideals. *Journal of pure and applied algebra*, (113):219–227, 1996.

[2] J. Cederquist and Th. Coquand. Entailment relations and distributive lattices. *To appear in the proceedings of Logic Colloquium 98*, 1999.

[3] M. Coste, H. Lombardi, and M.F. Roy. Dynamical method in algebra: Effective nullstellensätze. *Preprint*, 1996.

[4] E. Dobbs, R. Fedder, and M. Fontana. Abstract riemann surfaces of integral domains and spectral spaces. *Ann. Mat. Pura Appl. (4) 148*, pages 101–115, 1987.

[5] H.M. Edwards. The genesis of ideal theory. *Arch. Hist. Ex. Sci.*, pages 321–378, 1980.

[6] H.M. Edwards. *Divisor Theory.* Birkauser Boston, 1990.

[7] L. Espanol. Le spectre d'un anneau dans l'algbre constructive et applications la dimension. *Cahiers Topologie Geom. Differentielle 24*, pages 133–144, 1983.

[8] L. Espanol. Dimension of boolean valued lattices and rings. *J. Pure Appl. Algebra*, pages 223–236, 1986.

[9] L. Espanol. The spectrum lattice of baer rings and polynomials. *Categorical algebra and its applications (Louvain-La-Neuve, 1987)*, pages 118–124, 1988.

[10] K.F. Gauss. *Disquisitiones Arithmeticae.* 1802.

[11] J. König. *Einleitung in die allgemeine Theorie der algebraischen Grössen.* Teubner, Leipzig, 1903.

[12] D. Scott. Completeness and axiomatiszability. *Proceedings of the Tarski Symposium*, pages 411–435, 1974.