Point-Free Topology and Sheaf Models

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This talk

Two examples from practice of constructive mathematics

(1) How to define Cantor space? Or the real interval $[0, 1]$?

(2) How to represent the algebraic closure of a field?

Classically, (1) is concerned with non countable sets, while the algebraic closure might be countable if we start from a countable field

In both cases, these sets are best represented not as "actual", "completed" totalities, but as "potential", "open" collections

The second example is connected with the history of the notion of *field* (Dedekind) versus *domain of rationality* (Galois, Abel, Kronecker)

This talk

The first example is inspired by the presentation of Cantor space in

Notes on Constructive Mathematics, P. Martin-Löf, 1968

The second example is a variation on two notes of A. Joyal

Les théorèmes de Chevalley-Tarski et remarque sur l'algèbre constructive 1975

La Logique des Topos 1982 (with André Boileau)

Cantor space

Cantor space X is classically the topological space $\{0, 1\}^{\mathbb{N}}$

Use $\alpha, \beta, \ldots$ for points of this space

Use $\sigma, \tau, \ldots$ for *finite* binary sequence

Finite sequences represent basic neighbourhoods of X

Cantor space

An open set can be seen as a predicate V on finite sequences

We assume V to be monotone: $V(\sigma')$ if $V(\sigma)$ and σ' extends σ

Write $\overline{\alpha}(n)$ the finite sequence $\alpha(0)\ldots\alpha(n-1)$

$\alpha\epsilon V$ means $\exists_n V(\overline{\alpha}(n))$

Cantor space

How to define that V *covers* X?

First attempt

$\forall_{\alpha}\alpha\epsilon V \ = \ \forall_{\alpha}\exists_{n}V(\overline{\alpha}(n))$

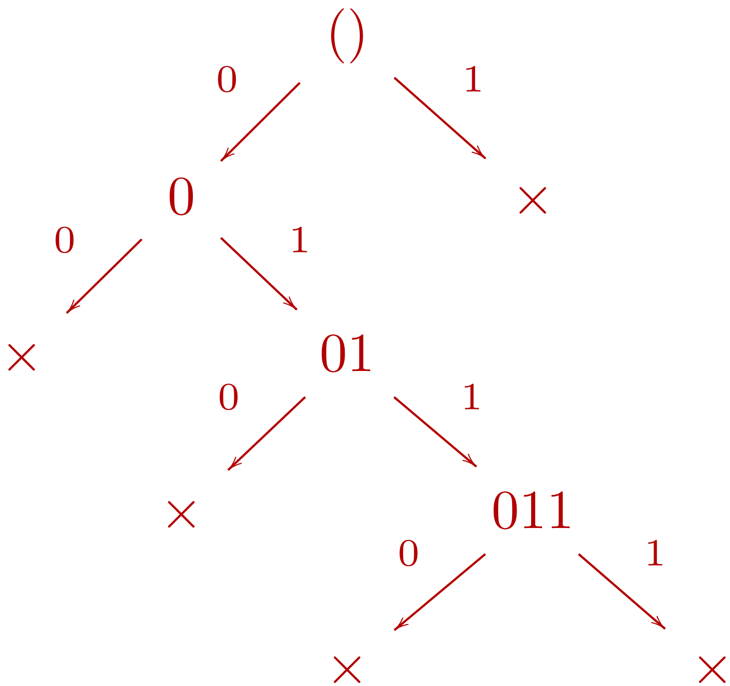
Brouwer formulated this as: V is a *bar*

V bars $X \ =_{def} \ \forall_{\alpha}\exists_{n}V(\overline{\alpha}(n))$

Whenever we make a sequence of successive choice $\alpha(0), \alpha(1), \ldots$

eventually we hit the bar V

Example of a bar



Brouwer's Fan Theorem

Brouwer's Fan Theorem can then be stated as the implication

V bars $X \;\rightarrow\; \exists_N \forall_\alpha V(\overline{\alpha}(N))$

or *any bar is a* uniform *bar*

This holds classically: compactness of Cantor space

Brouwer's Fan Theorem

One possible formulation would then be

$$\forall_\alpha \exists_n V(\overline{\alpha}(n)) \;\rightarrow\; \exists_N \forall_\alpha V(\overline{\alpha}(N))$$

However, this does *not* hold constructively

Kleene defined explicitely V such that

(1) one can find arbitrary long finite sequences not in V

(2) any *computable* α is barred by V

Intuitively, reduces to the fact that we can enumerate computable functions

Brouwer's Fan Theorem

In constructive mathematics, such as developed by Bishop, one *cannot* prove

V bars $X \;\rightarrow\; \exists_N \forall_\alpha V(\overline{\alpha}(N))$

if we identify V bars X with $\forall_\alpha \exists_n V(\overline{\alpha}(n))$

One intuition: we want to express that for *any* sequence of successive choices $\alpha(0), \alpha(1), \alpha(2), \ldots$ we will *eventually* get a number n such that $V(\overline{\alpha}(n))$ but we do *not* want to restrict this sequence of choices to one given by a law

Brouwer's Fan Theorem

$$\forall_\alpha \exists_n V(\overline{\alpha}(n)) \ \rightarrow \ \exists_N \forall_\alpha V(\overline{\alpha}(N))$$

This is also not provable in dependent type theory, where we analyse the meaning of universal quantification

The problem is that we consider X as a set of its points

We do not force in the formalism $\mathbb{N} \rightarrow \{0,1\}$ to only consist of computable functions

Not clear how to change this meaning to make this implication valid (maybe some hints toward the end of the talk)

What is a bar?

Solution in constructive mathematics: to define V bars X in a *point-free way*

We define V bars σ inductively

(1) V bars σ if $V(\sigma)$

(2) V bars σ if V bars $\sigma 0$ and V bars $\sigma 1$

Define then V bars X to mean V bars $()$

Then we can prove V bars $X \rightarrow \exists_N \forall_\alpha V(\overline{\alpha}(N))$

What is a bar?

The inductive definition of V bars $()$ is the correct formal expression of a universal quantification over *all* sequences, not necessarily given by a law

More generally V bars σ describes in a correct way Cantor space as a topological space

This is a motivation for *point-free* topology, where one describes a space not as a collection of points, but in term of its basic open and a covering relation V bars σ described inductively

What is a bar?

"We consider the arguments given by Brouwer 1927 in the proof of the bar theorem, rather as an intuitive analysis justifying the definition we have adopted of what it means for a Π^1_1 statement to hold."

P. Martin-Löf *Notes on Constructive Mathematics*, 1968

See also the end of W. Veldman *Intuitionism, an inspiration?*, 2021

The continuum

The same analysis can be done for $[0, 1]$

The fact that such a space cannot be considered as the set of its points is stressed in a forceful way by H. Weyl

Über due neue Grundalgenkrise der Mathematik, 1921

The continuum

This point-free approach was also adopted by Lorenzen (for Cantor-Bendixson)

Logical reflection and formalism, 1958

"This new analysis will never have to use the "naive" concept of a set, and so I would like to call it a "critical" analysis... In a critical analysis, however, one would have to avoid not only in the results, but also in the proofs, any reference to the naive concept of relation or function. In work on recursive analysis this latter is often not done."

What is a space?

What is a space?

Defined in term of basic open and covering relation

If we write $\alpha\epsilon\sigma$ for $\exists_n \sigma = \overline{\alpha}(n)$ and $\alpha\epsilon V$ for $\exists_n V(\overline{\alpha}(n))$

V bars σ is the "correct" formulation of

$\forall_\alpha\ \alpha\epsilon\sigma \rightarrow \alpha\epsilon V$

What is a space?

"Phenomemological" description of the space in term of finite syntactical elements, without assuming the space to be a collection of its points

Strong analogy with thermodynamics which describes phenomena without assuming a reduction in terms of molecular details or microscopic processes

What is a space?

The same approach works in *algebra* for representing the *Zariski spectrum* of a ring R

Distributive lattice generated by symbols $D(a)$ and relations

$$D(0) = 0 \qquad D(1) = 1 \qquad D(ab) = D(a) \wedge D(b) \qquad D(a+b) \leqslant D(a) \vee D(b)$$

Such a description would not be possible in terms of points (prime ideals)

Algebraic closure

Each syntactical/finite element σ can be thought of as a possible finite piece of information/knowledge about a "generic" sequence α

We have the basic covering $\sigma \lhd \sigma 0, \sigma 1$ which describes the topology of Cantor space X

The notion of *site* introduced by Grothendieck generalizes this situation

It can be used to describe constructively the algebraic closure of a field F

Constructive algebra

Algebraic closure in constructive mathematics??

The problem is more basic than use of Zorn's Lemma

We cannot decide if a given polynomial in $F[X]$ is irreducible or not

Theorem: *The sentence* $(\forall_x \; x^2 + 1 \neq 0) \vee \exists_x \; x^2 + 1 = 0$ *is not provable*

Algebraic closure

F field

A finite approximation of the algebraic closure of F can be thought of as a *triangular* algebra

Definition: *A F-algebra is triangular if it can be obtained from F by a sequence of (formal) monic separable extensions*

Monic: leading coefficient is 1

P separable: we have $AP + BP' = 1$ "all roots are simple roots"

Algebraic closure

Example: $\mathbb{Q}[x]$ where $x^2 = 3$ and then $\mathbb{Q}[x, y]$ where $y^3 + xy + 1 = 0$

Theorem: *If R is triangular then $R = R/(a) \times R[1/a]$ for all a in R*

Furthermore $R/(a)$ and $R[1/a]$ are products of triangular algebras

What is important is that the computations never involve irreducibility tests, only computations of g.c.d. of polynomials

Algebraic closure

The triangular algebra plays the role of finite binary sequences

Basic covering? We define a *site*, notion introduced by Grothendieck

Objects: triangular F-algebra

Maps: maps of F-algebra

Coverings:

$R = R_1 \times \cdots \times R_m$ for instance $R = R/(a) \times R[1/a]$

$R \rightarrow R[X]/(P)$ with P separable monic polynomial

Site

What is a *sheaf* over this site?

We should have $L(R)$ set for each R with transition maps $L(R) \to L(S)$

(1) $L(R) = L(R_1) \times \cdots \times L(R_m)$ if $R = R_1 \times \cdots \times R_m$

(2) sheaf condition for $L(R) \to L(R[X]/(P))$

Algebraic closure

In the topos model over this site, we can consider the presheaf

$L(R) = Hom(F[X], R)$

(Note that $F[X]$ is not in the base category, not being triangular)

$L(R)$ can be identified with the set of elements of the algebra R

Theorem: L *is actually a* sheaf *and is the (separable) algebraic closure of* F

Algebraic closure

In this sheaf model, L satisfies the following axioms

$1 \neq 0 \qquad\qquad \forall_x \;\; x = 0 \vee \exists_y \; (xy = 1)$

$\forall_{x_1 \dots x_n} \exists_x \quad x^n + x_1 x^{n-1} + \cdots + x_n = 0$

$\forall_x \bigvee_P \; P(x) = 0$

where the disjunction is over all monic separable polynomials P in $F[X]$

So L is the algebraic closure of F

But L is *not* defined as a set

Algebraic closure

L is not given as a *set* but as a collection of sets $L(R)$

Each triangular algebra R can be described by a finite syntactical object

At any stage of knowledge R, we only have access to a finite approximation of the algebraic closure

Algebraic closure

This is compatible with the description of Galois, Abel, Kronecker in term of adjoining a finite number of algebraic quantity

This point is stressed by Harold Edwards, in *Divisor Theory*

"It is usual in algebraic geometry to consider function fields over an *algebraically closed field*-the field of complex numbers of the field of algebraic numbers-rather than over $\mathbb{Q}$. In the Kroneckerian approach, the transfinite construction of algebraically closed field is avoided by the simple expedient of adjoining new algebraic numbers to $\mathbb{Q}$ as *needed*."

Algebraic closure

"The necessity of using an algebraically closed ground field introduced-and has perpetuated for 110 years-a fundamentally *transcendental* construction at the foundation of the theory of *algebraic* curves. Kronecker's approach, which calls for adjoining new constants algebraically as they are needed, is much more consonant with the nature of the subject."

in *Mathematical Ideas, Ideals and Ideology*, 1992

Algebraic closure and computations

“Clearly, a numerical extension of a numerical extension is a numerical extension. Moreover, given two numerical extension L and L' of the same function field K, there is a third numerical extension L'' of K which contains subextesions isomorphic to L and L'.”

The algebraic closure of K is conceived as an open, never finished totality that we can only access through its finite approximations

Algebraic closure

This is to be contrasted with Dedekind-Weber's approach who used for L the *set* of all complex numbers (or of all algebraic number)

Dedekind actually defined a *field* as a *subset* of the set $\mathbb{C}$

To use the set $\mathbb{C}$ as a complete totality was important for Dedekind and Weber, but not for Kronecker, not being "consonant with the nature of the subject"

The fact that Kronecker never considered $\mathbb{C}$ as an actual totality is missed in Bourbaki's historical notes who writes that Kronecker was concerned with ideals of $\mathbb{C}[X_1, \ldots, X_n]$

Algebraic closure and computations

In this approach the algebraic closure is never given in its totality but suitable finite approximations are unfolded by doing computations

This model is *effective*

This description was motivated by the dynamical technique introduced in computer algebra by Dominique Duval (1985)

cf. Teo Mora's book

Solving Polynomial Equation Systems: the Kronecker-Duval Philosophy

Example of computation

E.g. Abhyankar's proof of Newton-Puiseux Theorem

Algebraic Geometry for Scientists and Engineers

course notes taken by Sudhir Ghorpade

Th. C. and Bassel Mannaa *A sheaf model of the algebraic closure*, 2014

For instance, given an equation $y^4 - 3y^2 + xy + x^2 = 0$ find y as a formal serie in x (in general $x^{1/n}$)?

The coefficients of this power serie have to be in an algebraic extension of $\mathbb{Q}$

Algebraic closure

We first prove that theorem assuming an algebraic closure of $\mathbb{Q}$

We need to consider *structures* we can build from L, in this examples $L((X))$

Theorem: $\cup_n L((X^{1/n}))$ *is separably closed*

Hence for $y^4 - 3y^2 + xy + x^2 = 0$ we can describe y as a formal serie in x with coefficients in L

Algebraic closure

Since this interpretation is *effective*, we can compute y in $y^4 - 3y^2 + xy + x^2 = 0$ as a formal serie in x in a triangular algebra $\mathbb{Q}[a, b]$ with $a^2 = 13/36$ and $b^2 = 3$

This finite extension is *created* or *actualized* by the computation

The algebraic closure is only given in a *potential* way

Algebraic closure

If F is countable there exists a construction of the algebraic closure of F

cf. *A Course on Constructive Algebra*, by Mines, Richman and Ruitenburg

But this involves a *non canonical* enumeration

So even in the countable case, this description in term of a potential totality seems both conceptually and computationally preferable

The classical argument involves a non canonical enumeration or Zorn's Lemma

Field as an actual totality

At around the same time (1897), Hensel introduced $\mathbb{Q}_p$

Yet another example of field as an actual totality

Algebraic p-adic numbers

F. Kuhlmann. H. Lombardi, H. Perdry

Dynamic computations inside the algebraic closure of a valued field, 2003

What is going on?

This way to build the algebraic closure is a special case of a general way to build the *classifying topos* of a geometric theory

The simplest example: theory of an *infinite* set

$\forall_{x_1,\dots,x_n} \exists_x \; x \neq x_1 \wedge \cdots \wedge x \neq x_n$

The site is the following

objects are *finite* sets, maps are arbitrary functions

basic covering $X \to X, x$

Classifying topos

Yet another example is the theory

$$\neg(x < x) \qquad x < y \wedge y < x \rightarrow x < z \qquad \exists_y \ (x < y)$$

which does not have any finite model

The classifying topos provides a finitistic model theoretic proof of consistency of this theory

Th. C. *A completness proof for geometric logic*, 2004

What is going on?

This is reminiscent of some remarks in *Formalism 64*, A. Robinson

"I was less definite in dicussing the logic which applies to systems of unbounded extent, i.e. which are potentially infinite. I now wish to suggest that for these a form of Modal Logic may be appropriate."

Appendix *A Notion of Potential Truth*

"It corresponds to the intuitive idea that an existential sentence is potentially true in a given structure if we can find an element that satisfies it by extending the structure far enough in the right direction."

Algebraic closure

Grothendieck's idea was that a sheaf model can be thought of as a new "frame" where one can develop mathematics

If A and B are sheaves, we can form $A \times B$ and $A \to B$ as sheaves

We can develop mathematics in this new "frame", e.g. define what is a group, what is a field, and so on

What is going on?

The algebraic closure of F may not exist in $Sh(1)$ the usual frame of sets, but may exist in $Sh(S)$ the frame of sheaves over S

J.L. Bell, *From Absolute to Local Mathematics*, 1988

Parallel with physics

Let S be a "space" (given as a point-free space or by a Grothendieck site)

Canonical map $f : S \to 1$ and $f^* : Sh(1) \to Sh(S)$

f^* allows us to change the frame of reference

What is going on?

Bell: This is like change of *reference frames* in physics

Each frame has it own notion of functions

E.g. there may be more functions $\mathbb{N} \to \{0, 1\}$ in $Sh(S)$ then in $Sh(1)$

V bars σ described in $Sh(1)$ captures the fact that in any possible sheaf model $Sh(S)$ we have $\forall_\alpha \exists_n V(\overline{\alpha}(n))$

What is going on?

In $Sh(1)$ we have the field F

It may not have an algebraic closure in $Sh(1)$

In $Sh(S)$ the field F has a (separable) algebraic closure!

Conclusion

Constructive mathematics contains examples of collections that are given only in a *potential* way

Not only uncountable collections such as Cantor space of $[0, 1]$, but also collections that classically may be countable, e.g. algebraic closure of a field

One can argue that this description is more consonent with the nature of what is going on mathematically

This description is also “dynamic” with e.g. an open notion of binary functions

I found it interesting that the notion of *site*, and of *classifying topos*, introduced by Grothendieck, seems to represent well Kronecker’s approach