Constructive Presheaf Models of Univalence

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Motivation

Symbolic representation of “spaces”

We want to represent spaces up to homotopy

E.g. the real lines and a point are identified

Mathematicians have designed a sophisticated framework to address this question: the notion of *Quillen Model Structure*

Quillen Model Structure

A QMS on a given category consists of 3 classes of maps

fibrations F *cofibrations* C *equivalences* W

We write $TF = W \cap F,\ TC = W \cap C$

(1) C, TF and TC, F should define a weak factorization system

(2) W should satisfy the "three-out-of-two" conditions

We consider in this talk only model structures on presheaf categories

Each QMS defines its own notion of "space"

Presheaf models of univalence

Why using *presheaves* to represent *spaces*?

Eilenberg and Zilber 1950 *Semi-Simplicial Complexes and Singular Homology*

An intuition is that the objects in the base category are given "shapes"

A presheaf is obtained by glueing together these basic shapes

So the intuition is *geometrical*

This is to be compared with the use of (pre)sheaves in logic, Beth 1954 or Kripke 1958, there the intuition is *temporal*

Quillen Model Structure

There is a notion of *Quillen functor* between two QMS

This is a property of a given functor between the two underlying categories

One can define when such a functor is a *Quillen equivalence*

Such a Quillen equivalence can be thought of as an evidence that the two QMS describe the same notion of spaces

Quillen Model Structure

There is a QMS on topological spaces Top

We have to define a QMS "purely symbolically", i.e. a category with a QMS

A functor G from this category to Top

This functor should be a Quillen equivalence

If $G(X)$ and $G(Y)$ are equivalent in Top then X and Y should be equivalent

(in our case, all objects will be cofibrant)

What has all this to do with computer science?

It has been noticed that there are strong analogies between the structure of spaces up to homotopy and highly modular structures that appear in mathematics

the intuition appeared that ∞-groupoids should constitute particularly adequate models for homotopy types, the n-groupoids corresponding to truncated homotopy types (with $\pi_i = 0$ for $i > n$)

Grothendieck, *Sketch of a program*, 1984

Cohomology

We illustrate this point by one example

Definition of cohomology groups

Cohomology

Let M/X the model of type theory of families of types over X

If A is an abelian group one can define $H^1(X, A)$ as the *groupoid* of A-torsors in M/X

The usual cohomology group is best seen as a *set truncation* of this groupoid

E.g.: The groupoid of $\mathbb{Z}$-torsors is equivalent to S^1 and the set truncation of $H^1(1, \mathbb{Z})$ is trivial

Over M/S^1 we have the helix which is a non trivial $\mathbb{Z}$-torsor

The groupoid $H^1(S^1, \mathbb{Z})$ has a non trivial set truncation

Cohomology

This can be extended to 2-groupoids to describe $H^2(X, A)$

Over S^2 the Hopf fibration shows that $H^2(S^2, \mathbb{Z})$ is non trivial

In general, we define $H^n(X, A)$ as the set truncation of $X \to B^n(A)$

Cohomology

These notions are surprisingly well expressed in the setting of dependent types extended with homotopy theoretic features

Notion of "existence unique up to unique isomorphism"

In this setting, this is replaced by a general and uniform method: define a type of solutions of a problem and show that this type is *contractible*

For this we can replace freely mere existence by explicit existence

We can use the principle of "unique choice"

Cf. The "Symmetry book" started at the CAS special year in Oslo 2018-2019

Cohomology

In general it is essential that the formal system we are using can represent such higher structures

Not only set like structure, like groups, rings, ...

Since higher structures and spaces up to homotopy seem to be closely connected (Grothendieck) one may try to use this connection to understand how to design a formal system for representing higher structures

Rather unexpected that it is related to λ-calculus!

What has all this to do with computer science?

Voevodsky used the QMS on *simplicial sets* to build a model of the univalence axiom, which expresses a strong form of *extensionality* (modularity) in dependent type theory

D. Kan *A combinatorial definition of homotopy groups*, 1958

The QMS on Kan simplicial sets is obtained in a highly *non effective* way

Hence this model of type theory does not help directly to make sense of the univalence axiom

Presheaf models of univalence

On the other hand, recent works have shown how to build model of univalent type theory as “inner” model inside a rather large class of presheaf models in a constructive way

Presheaf models of univalence

All we need is a presheaf (usually representable) which plays the rôle of an interval

It has two global distinct element $0, 1$

We however need a more technical condition: this presheaf has to be *tiny*

A presheaf X is *tiny* if the functor $A \longmapsto A^X$ has a right adjoint

This is satisfied if $\mathbb{I}$ is representable and the base category is closed by products with $\mathbb{I}$: we have then $A^{\mathbb{I}}(K) = A(K \times \mathbb{I})$

Type theoretic construction of Model Structures

These models can be seen as (non standard) *homotopy theoretic* model extension of type theory

This terminology is justified since we can reverse the direction

QMS on Simplicial Sets $\rightarrow$ model of univalent type theory

to

models of univalent type theory $\rightarrow$ QMS on presheaf categories

Type theoretic construction of Model Structures

So not only we can build model of univalence

But furthermore we can use the fibrant universe to build a QMS

A natural question is then: can we describe Top in this way?

QMS on Presheaves

We define C to be the class of *monomorphisms* (technically, we need locally of decidable image)

We define F to be the class of maps having a form of path lifting properties

We define the classes TC and TF by orthogonality

There is then no choice for the definition of W

All this has been formalised: thesis of Simon Boulier (December 2018)!

Type theoretic construction of Model Structures

One key point is to have a *fibrant* universe (of fibrant types)

Furthermore these QMS satisfy

-Frobenius (and right properness: W is preserved by pullbacks along fibrations)

-Equivalence Extension Property

-Fibration Extension Property (a.k.a. "Joyal" property)

New class of *complete* Cisinski model structure

Type theoretic construction of Model Structures, some features

-These models can be developed in a *constructive* meta theory

-They can be developed using the internal language of presheaf categories (model of dependent type theory),

-They have been *formalised* (in Agda)

Type theoretic construction of Model Structures, some features

There is a correspondance between these proofs in the internal language and "diagrammatic" proofs

So far, there seems to be a "speed-up" phenomenon: a proof of a few lines in the internal language corresponds to a proof taking several pages when formulated in diagrams (for instance, the proof of the Frobenius property)

Fibration on presheaves

A a dependent type on Γ

A is *fibrant* if for any γ in $\Gamma^{\mathbb{I}}$ and any partial section u in $\Pi(x:\mathbb{I})A\gamma(x)$ only defined on a truth value ψ and $x_0:\mathbb{I}$ and $u_0:A\gamma(x_0)$ compatible with u, we can find $c_A\ \gamma\ (\psi \to u)\ x_0\ u_0$ total section which extends u

Type theoretic construction of Model Structures, some features

A special case is the *transport* or path lifting operation t_A

$t_A\ \gamma\ x_0\ a_0 \quad = \quad c_A\ \gamma\ (\bot\!\!\rightarrow a)\ x_0\ a_0$ this extension operation

A section defined only at a given point of $\mathbb{I}$ can be extended to a total section

It is then possible to show that all type forming operations can be lifted in order to produce corresponding structure preserving operations

Fibration on presheaves

The proof that if A is fibrant and B fibrant on $\Gamma.A$ then $\Pi\ A\ B$ is fibrant on Γ is then one line

It corresponds to the Frobenius property

The proof of this property takes several pages when expressed with categorical diagrams

Type theoretic construction of Model Structures, some features

Internally, most of the model can be described in this way

What cannot be described in this way however, is the existence of a universe classifying fibrations *with a given structure*

This uses in a crucial way that the interval is tiny

-Δ^1 is *not* tiny, so this method does not apply (so far) to simplicial sets

Cartesian Cubical Sets

In particular, this works for *cartesian* cubes

Base category: $\{0,1\}^n$ the maps $\{0,1\}^n \rightarrow \{0,1\}$ are only the constants and the projections ($n+2$ maps)

We get in this way a Quillen model structure on the *same* presheaf category mentionned by Grothendieck in "Pursuing Stacks" for building "in a sense the simplest test model category"

Do we get the same QMS?

Cartesian Cubical Sets

Cartesian cubes are interesting *classically*, since the base category $\square$ is *generalized Reedy*

This means that we can do arguments/constructions by induction on the dimension, but we have to take into account the symmetries

By opposition, simplicial sets are "rigid": no symmetries and we can directly do induction on the dimension

Cartesian Cubical Sets

We say that a presheaf F (non necessarily fibrant) is weakly contractible if the canonical map $F \to 1$ is an equivalence

Christian Sattler found out that the quotient of a square by swapping is *not* weakly contractible for this QMS

So geometric realization cannot be a Quillen equivalence, and the QMS we have defined is *not* equivalent to Top via geometric realization

New: this issue is solved by imposing the further property of *equivariance*!

Equivariance

Filling operation: lifting property w.r.t. generalized boxes

Generalized box: given A subobject of B and b a point of the interval $\mathbb{I}$ a box is the subobject of $B \times \mathbb{I}$ determined by $B \times b \cup A \times \mathbb{I}$

We generalize the point inclusion to: a tuple of points in $\mathbb{I}^n$

The filling has to be equivariant w.r.t. any permutation of $\mathbb{I}^n$

In the cartesian cubical set, the permutations of $\mathbb{I}^n$ are exactly the automorphisms of $\mathbb{I}^n$

Equivariance

Here is the internal description

Given A dependent type over Γ, we consider $\gamma : \mathbb{I}^n \to \Gamma$

We write $X = \mathbb{I}^n$

A fibration structure extends any partial section of $\Pi(x : X)A\gamma(x)$ defined only for $x = x_0$ or ψ to a total section section and extends this in an equivariant way

If we write $c_A\ (\psi \to a)\ x_0\ a_0$ this extension operation

We should have

$$c_A\ \gamma\ (\psi \to a)\ (\sigma x_0)\ a_0\ (\sigma x) \quad = \quad c_A\ \gamma\sigma\ (\psi \to a\sigma)\ x_0\ a_0\ x$$

Cartesian Cubical Sets

We can then adapt a formalisation of the non equivariant model and check that all structure preserving operations can be extended in an equivariant way

The main facts (in particular the ones that imply that the universe of fibrant types is fibrant) have been checked formally in Agda (Evan Cavallo, CMU)

Cartesian Cubical Sets

For this QMS, all quotients $\mathbb{I}^n/G$, where G finite group, are weakly contractible

Indeed, all maps $1 \rightarrow X$ are (by transport) trivial cofibration

We have

$t_A\ \gamma\ (\sigma x_0)\ a_0\ (\sigma x) = c_A\ \gamma\sigma\ x_0\ a_0\ x$

Cartesian Cubical Sets and Generalized Reedy Property

We can *classically* build a section (Excluded-Middle) of

$$X + \neg X \qquad (X : U,\ h : \mathsf{isProp}\ X)$$

building it by *induction on dimension*

In particular, in this model Bool is classically equivalent to $\mathsf{hProp}(U)$!

Cartesian Cubical Sets and Generalized Reedy Property

Classically, we have

The triangulation map cSet → sSet *is a Quillen equivalence* (Christian Sattler)

To summarize: we get a model of type theory, which classically validates Excluded-Middle, and whose associated Quillen model structure is via geometric realization classically a Quillen equivalence

Cartesian and Dedekind cubical sets

There is another candidate for representing spaces

The base category is the opposite of finitely presented distributive lattices

There is a natural geometrical realisation

It is not known whether or not this is a Quillen equivalence

(pre)Sheaf models

This actually works for a large class of presheaf models: all we need is a tiny interval object

In particular, we can build in this way a QMS on *cubical presheaves*, i.e. presheaves over $\mathcal{C} \times \square$ where $\mathcal{C}$ is any small category

We define a new interval $\tilde{\mathbb{I}}(X, J) = \mathbb{I}(J)$ which is still tiny

$F^{\tilde{\mathbb{I}}}(X, J) = F(X \times \mathbb{I}, J)$

Do we get the "right" notion of equivalence (pointwise equivalence)?

Example 1

Cubical presheaves over the poset $0 \leqslant 1$

In this case $\mathsf{D}(F)$ can be seen as an exponential F^C for some C

Any presheaf is already modal: we don't need to localize

A presheaf is exactly a fibration $F_1 \rightarrow F_0$ of cubical sets

Example 2

Cubical presheaves over the poset $X_0 \geqslant X_1 \geqslant X_2 \geqslant \ldots$

In this case, we need to localise

Example 3

Model of *parametrised pointed types*

Cubical presheaves over category: X with an idempotent endomap f