CONSTRUCTIVE TOPOLOGY AND COMBINATORICS

Thierry Coquand

Abstract.

We present a method to extract constructive proofs from classical arguments proved by topogical means. Typically, this method will apply to the nonconstructive use of compactness in combinatorics, often in the form of the use of König's lemma (which says that a finitely branching tree that is infinite has an infinite branch.) The method consists roughly of working with the corresponding point-free version of the topological argument, which can be proven constructively using only as primitive the notion of inductive definition. We illustrate here this method on the classical "minimal bad sequence" argument used by Nash-Williams in his proof of Kruskal's theorem. The proofs we get by this method are well-suited for mechanisation in interactive proof systems that allow the user to introduce inductively defined notions, such as NuPrl, or Martin-Löf set theory.

Introduction and description of the method.

The idea of point-free topology seems to go back to Whitehead (around 1910) and has been presented by Russell (see Russell 1914, 1936 and Wiener 1914). The problem is to relate the mathematical notion of points and what is given to us as observable by our sense data. The point-free approach is then a kind of reverse of the usual set-theoretical approach: instead of defining an observable as a collection of points, the notion of observable is taken as primitive, and a point is defined as a collection of observable. It is hoped then that it will be easier to relate a mathematical result to its potential application, since its formulation is using the concrete notion of observable. The main application of point-free topology for Russell seems indeed to have been an analysis of the use of mathematics in physics. For instance, the notion of instants of time is analysed in terms of the notion of events. One problem that Russell mentions is the need of using the axiom of choice in order to construct points.

This idea of point-free topology has been revived recently in the framework of the theory of locale, see Johnstone 1981, and Vickers 1989. It is instructive to see how the problem of the use of the axiom of choice in the construction of points of a locale, which, as we saw, was mentionned by Russell, is analysed in this framework. As described by Johnstone 1981, what happens is that it is actually in many ways more convenient to work directly with locales, and simply to forget about points (Johnstone traces this idea back to Isbell).

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The situation is particularly interesting from a constructive point of view. The point-free approach to topology can be rightly described as constructive topology, and many constructions that need the axiom of choice or highly nonconstructive arguments in terms of points get a direct constructive proof when stated in a point-free way. A typical example is the theorem of Tychonoff, which is classically equivalent to the axiom of choice, but whose point-free version is provable without this axiom, see Johnstone 1981.

In this paper, we want to present point-free topology as a way to get constructive contents from classical proofs. An early application of this method appears in De Bruijn and Van der Meiden 1967, where sufficiently large closed ideals are used instead of maximal ideals in order to give a version of Wiener's proof on absolutely convergent series which does not use the axiom of choice. As noticed by the authors, we get by this point-free approach a constructive development entirely parallel to the classical approach, such that it is possible at every stage to reach the corresponding stage in the usual theory by an application of the excluded middle and the axiom of choice.

The point-free approach has been used also in domain theory, see Martin-Löf 83, Abramsky 87, leading to a generalisation of the predicate transformer method of Dijkstra. In Martin-Löf 83, it is particularly clear that the point-free spaces are naturally described in the framework of inductive definitions. This seems to be a general phenomenon, which is particularly interesting for mechanisation of proofs, that both the basic notions and proofs of point-free topology get a natural formulation with inductive definitions. For instance, in Coquand 91, it is shown that Tychonoff's theorem has a direct inductive point-free proof.

In this paper, we present yet another application of the point-free approach. It concerns combinatoric statements, that are proved using topological argument. Here, we present a point-free analysis of a topological formulation of the classical "minimal bad sequence" argument used by Nash-Williams 1963 in his proof of Kruskal's theorem. The result follows the general pattern: while the classical statement is highly nonconstructive and needs the axiom of choice in its proof, its point-free version gets a direct inductive proof.

1. An open induction principle.

Let X be a set, and $W(X)$ be the set of finite words over X, $S(X)$ the set of infinite words (or streams) over X. We can consider $W(X)$ as a transition system, letting $u \to v$ mean that v can be written ux, $x \in X$. We say that a subset $B \subseteq W(X)$ is **monotone** iff $u \in B$ and $u \to v$ imply $v \in B$. If $B \subseteq W(X)$ is monotone we let $U(B) \subseteq S(X)$ be the set of infinite words $\alpha_1\alpha_2\ldots$ such that there exists p such that $\alpha_1 \ldots \alpha_p \in B$. We say that $U \subseteq S(X)$ is **open** iff it can be written $U = U(B)$ for one monotone $B \subseteq W(X)$.

For instance, let $\prec$ be a binary relation on X, and let $U \subseteq S(X)$ be the set of infinite words $\alpha_1\alpha_2\ldots$ such that there exist i, j such that $i < j$ and $\alpha_i \prec \alpha_j$. Then U is open. To say that $\prec$ is a well-quasi-order on X is the same as to say that $U = S(X)$.

We recall that a relation $<$ on X is **well-founded** iff the following induction principle holds: if $M \subseteq X$ is such that $x \in M$ whenever $y \in M$ for all $y \in X$ such that $y < x$, then

M is equal to X. This implies that there are no infinite decreasing sequences. The converse holds intuitionistically only if we consider arbitrary choice sequences.

Classically, we use rather the following equivalent definition, that X is well-founded iff all non-empty subset of X has a minimal element for the relation $<$.

We suppose that $<$ a well-founded relation over a set X. We extend $<$ alphabetically on $W(X)$ by letting $x_1 \dots x_p < y_1 \dots y_q$ mean that $p \leq q$ and $x_1 = y_1, \dots, x_{p-1} = y_{p-1}$, $x_p < y_p$. Notice that if $u < v$, then $u < vx$ for all $x \in X$.

We extend next the relation $<$ to $S(X)$: we let $\alpha < \beta$ between $\alpha = \alpha_1\alpha_2 \dots$ and $\beta = \beta_1\beta_2 \dots$ mean that there exists p such that $\alpha_1 \dots \alpha_p < \beta_1 \dots \beta_p$. The relation $<$ on infinite words is not well-founded in general. For instance, if X is the set $\{0, 1\}$ with the ordering $0 < 1$, we have the infinite decreasing sequence $1111 \cdots > 0111 \cdots > 0011 \cdots > 0001 \dots$.

However, we can still express the following induction principle, where we restrict ourselves to open properties.

Theorem. *If $U \subseteq S(X)$ is open, and $\alpha \in U$ whenever $\beta \in U$ for all $\beta < \alpha$, then U is equal to $S(X)$.*

This fact can be seen as a topological formulation of Nash-Williams minimal bad sequence arguments, and has been noticed by Raoult 1988.

The proof given in Raoult 1988 is however based on a general argument that uses Zorn's lemma. It is also possible to prove the theorem directly, following the minimal bad sequence argument, by reasoning classically and using the axiom of choice.

Here is a sketch of the classical reasoning. Notice how difficult it would be to formalise this argument completely.

If U is not equal to $S(X)$ there exists $\alpha \in S(X)$ not in U. The set of $x \in X$ such that there exists $\alpha \in S(X)$ not in U satisfying $\alpha_1 = x$ is thus non empty. Since $<$ is well-founded, there exists $x_1 \in X$ minimal such that there exists $\alpha \in S(X)$ not in U satisfying $\alpha_1 = x$. In the same way, we can find $x_2 \in X$ minimal such that there exists $\alpha \in S(X)$ not in U satisfying $\alpha_1 = x_1$ and $\alpha_2 = x_2$. Using the axiom of dependent choice, we build an infinite word $x_1x_2x_3 \dots$ such that, for each k, there exists $\alpha \in S(X)$ not in U extending $x_1 \dots x_k$. Since U is open, this infinite word $\alpha = x_1x_2 \dots$ is not in U. Furthermore, we have by construction that $\beta < \alpha$ implies $\beta \in U$. This contradicts the hypothesis on U.

We show in the next section that the point-free version of this theorem can be proved directly constructively, using only as primitive the notion of inductive definition. This is yet another example of a general fact, already noticed in the case of Tychonoff's theorem, that a point-free version of a topological theorem can be proved directly constructively.

2. A point-free formulation.

The point-free approach will consider only "observable elements" that is, only finite words. We will see now an infinite words $\alpha_1\alpha_2\ldots$ as a description of a sequence of successive choices $\alpha_1 \rightarrow \alpha_1\alpha_2 \rightarrow \alpha_1\alpha_2\alpha_3 \rightarrow \ldots$

A subset Q of $W(X)$ is **hereditary** iff $u \in Q$ whenever all v such that $u \rightarrow v$ are in Q. More generally, if $E \subseteq W(X)$, a subset $Q \subseteq W(X)$ is **hereditary on** E iff $u \in Q$ whenever $u \in E$ and $v \in Q$ if $u \rightarrow v$, $v \in E$. If $B \subseteq W(X)$, the hereditary closure E_B of B is the intersection of all hereditary subsets of $W(X)$ that contain B. This is a set-theoretical description of E_B. We can also consider E_B as an instance of an inductive definition, see Martin-Löf 71, and define E_B inductively by the introduction rule that $B \subseteq E_B$ and that E_B is hereditary. The operation from B to E_B correspons to the "eventually" operator used in temporal logic.

Notice that $v \in E_B$ mean classically that any infinite words that extend v is in $U(B)$.

It can be shown directly that E_B is monotone if B is monotone.

We can express in a point-free way when an open $U(B) \subseteq S(X)$ is the set of all infinite words. We say that a monotone $B \subseteq W(X)$ is a **bar** iff $E_B = W(X)$. For any subset $B \subseteq W(X)$, it can be checked classically, using the axiom of dependent choice, that B is a bar iff $U(B) = S(X)$.

More generally, if $B \subseteq W(X)$ is monotone and $E \subseteq W(X)$, we say that B is **a bar on** E iff the hereditary closure of $B \cap E$ in E is E, that is, if $F \subseteq E$ is such that $E \cap B \subseteq F$, and $x \in F$ whenever $x \in E$ and $y \in F$ if $x \rightarrow y$ and $y \in E$, then $E = F$.

Intuitively, a monotone $B \subseteq W(X)$ is a bar iff for any sequence of successive choice $u_1 \rightarrow u_2 \rightarrow u_3\ldots$, we have eventually $u_p \in B$. Likewise, a monotone $B \subseteq W(X)$ is a bar on $E \subseteq W(X)$ iff for any sequence of successive choice $u_1 \rightarrow u_2 \rightarrow u_3\ldots$, such that all u_k are in E, we have eventually $u_p \in B$.

In order to find a point-free equivalent of the open induction principle, let us first express it in intuitive terms that mention only observable terms. We start with a monotone subset $B \subseteq W(X)$ and we want to find a sufficient condition that implies that B is a bar.

The open induction principle says that, in order for this to hold, it is enough to have $\alpha \in U(B)$ whenever $\beta < \alpha$ implies $\beta \in U(B)$. The condition that $\beta < \alpha$ implies $\beta \in U(B)$ can be interpreted as follow: for all p, if $v < \alpha_1 \ldots \alpha_p$ and $\beta \in S(X)$ extends v, then $\beta \in U(B)$. This can still be expressed by saying that $v < \alpha_1 \ldots \alpha_p$ implies $v \in E_B$.

We then get by this analysis a statement that involves only observable terms. This is the following sufficient condition for B to be a bar. If we make a sequence of successive choices $\alpha_1 \rightarrow \alpha_1\alpha_2 \rightarrow \ldots$ in such a way that at each steps, $v < \alpha_1 \ldots \alpha_k$ implies $v \in E_B$, then eventually $\alpha_1 \ldots \alpha_p \in B$.

Let us state now formally the result of our analysis. Let $M(B)$ be the set of words $u \in W(X)$ such that $v < u$ implies $v \in E_B$.

Proposition. *E_B is hereditary on $M(B)$.*

Proof. Let $w \in M(B)$ be such that $wx \in E_B$ if $wx \in M(B)$. We show then $wx \in E_B$ for all $x \in X$ by well-founded induction on x. Indeed, if $wy \in E_B$ for all $y < x$, then $u \in E_B$ for all u such that $u < wx$, because in this case either $u < w$, which implies $u \in E_B$ because $w \in M(B)$, or $u = wy$ for some $y < x$, which implies $u \in E_B$ by hypothesis. Hence $wx \in M(B)$. This implies by hypothesis $wx \in E_B$. □

Corollary. *If B is a bar on $M(B)$, then B is a bar.*

Proof. If B is a bar on $M(B)$, then by the proposition, the empty word is in E_B. □

The content of this corollary is classically the same as the one of the open induction principle, hence the same as the one of the "minimal bad sequence"argument. It would be possible from it to deduce a purely inductive proof of Kruskal theorem. Observe that this statement refers only to observable terms, and the proof uses only the notion of inductive definition.

5. Conclusion.

A general remark is how a (classically minor) change in the formulation of a mathematical result can change its logical strength. The open induction principle for instance is classically directly equivalent to its point-free version. But its point-free version is proved directly whereas the version with points need the excluded middle and the axiom of choice to be proved. A similar remark appears already in Lorenzen 1958, where it is shown how a point-free version of Cantor-Bendixson is directly expressed, whereas the usual formulation needs very strong logical principle.

It may be interesting to analyse how this remark applies to statements analysed in reverse mathematics, typically to statements that are proved equivalent to weak König's lemma.

It would be also quite interesting to apply this point-free method in more complicated combinatorial statements proved by topological means. For instance, the topological proof of Van der Waerden theorem about arithmetical progression seems to call for a point-free theory of a continuous action of a group on a compact space.

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