ABELIAN l-GROUPS, GENERALISED REALS AND OPEN LOCALES

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Abstract. This is a summary of a discussion with Peter Schuster (march, 2003)

Introduction

To any abelian l-group with a strong unit, we associate a compact locale. In general the norm of an element is not a constructive real (Dedekind cut) but only a generalised real. If all elements are *normable*, then the associated compact locale is *open*. If the l-group is furthermore separable, then with dependent choices we can build points effectively.

1. Abelian l-group and compact space

Let R be an abelian l-group that is an abelian ordered group with a lattice operation $x \vee y$. Remarkable properties hold then in general: we have also an operation $x \wedge y$ defined by the law

$$x \wedge y + x \vee y = x + y$$

and the lattice $R, \wedge, \vee$ is distributive. We have also

$$n(x \vee y) = nx \vee ny,\ \ n(x \wedge y) = nx \wedge ny$$

and $0 \leq nx,\ n > 0$ implies $0 \leq x$. We can alaways embed R in a divisible group by adding formally elements $x/n,\ n > 0$, and we assume from now on that R is divisible. We write $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$.

We assume also that R has a strong unit $1 \geq 0$: for any $x \in R$ there exists n such that $x \leq n1$. We write often q instead of $q1$ for $q \in \mathbb{Q}$. Clearly we have $q \leq q' \in R$ if $q \leq q' \in \mathbb{Q}$. We assume that this is an equivalence: if $q \leq q'$ in R this holds also in $\mathbb{Q}$. We can consider

$$U_x = \{q \in \mathbb{Q} \mid (\exists r \in \mathbb{Q})\ |x| \leq r < q\}$$

The set U_x is almost like a upper Dedekind cut: we have

(1) if $q \in U_x$ there exists $q' < q$ such that $q' \in U_x$
(2) if $q \in U_x$ and $q \leq q'$ then $q' \in U_x$
(3) U_x is inhabited
(4) if $p < 0$ then $p < q$ for all $q \in U_x$

However we don't have in general that if $p < q$ then $q \in U_x$ or $p < r$ for all $r \in U_x$. If this is the case, we say that x is *normable*. It is clear that in this case U_x has a greatest lower bound which is a constructive real that will be written $||x||$.

We consider first the locale $Sp(R)$ defined by the generators $D(x),\ x \in R$ and the relations

(1) $D(a), D(-a) \vdash$
(2) $D(a) \vdash$ if $a \leq 0$
(3) $D(a+b) \vdash D(a), D(b)$
(4) $D(a \vee b) \vdash D(a), D(b)$
(5) $\vdash D(1)$

Proposition 1.1. *In $Sp(R)$ we have $D(a) \vdash D(b)$ whenever $a \leq b$. In particular $D(a) \vdash D(a^+)$. We also can prove*

$$D(a), D(b) \vdash D(a+b)$$
$$D(a), D(b) \vdash D(a \wedge b)$$
$$D(a^+) \vdash D(a)$$

Let P be the set of elements x that are ≥ 0. We let L be the distributive lattice $P, \wedge, \vee, \preceq$ with the preorder relation $x \preceq y$ defined by

$$(\exists n)\ x \leq ny$$

Theorem 1.2. *The map $m : a \longmapsto a^+,\ R \to L$ is an interpretation of the axioms $1, 2, 3, 4$ of the theory $Sp(R)$ in the distributive lattice L. Furthermore it is the* free *such interpretation. In particular, $a \vdash b$ is derivable in $Sp(R)$ iff $1 \wedge a^+ \preceq b^+$.*

The points of $Sp(R)$ are the prime ideals of R. The maximal ideals of R form another locale $Max(R)$ which is defined by the theory $Sp(R)$ extended with the infinitary axiom

$$D(a) \vdash \bigvee_{r>0} D(a-r)$$

Theorem 1.3. *$Max(R)$ is a compact completely regular locale which is a retract of $Sp(R)$. Any element $a \in R$ defines a localic map $\hat{a} : Max(R) \to \mathbb{R}$ and, via the map $a \longmapsto \hat{a}$, R is a dense sub l-group of $C(Max(R))$ and we have $r \in U_a$ iff $\vdash D(r-a)$ in $Max(R)$. More generally, $a \vdash b$ is derivable in $Max(R)$ iff $1 \wedge (a-r)^+ \preceq b^+$ for all $r > 0$*

2. When is $Max(R)$ open?

Theorem 2.1. *If all elements of R are normable then $Max(R)$ is an* open locale *with the positivity predicate $Pos(a)$ defined by $||a^+|| > 0$.*

Proof. Notice that $Pos(a \vee b)$ implies $Pos(a) \vee Pos(b)$. Indeed we have $(a \vee b)^+ = a^+ \vee b^+$. Since

$$0 < ||a^+ \vee b^+|| \leq ||a^+|| + ||b^+||$$

this follows from the valid implication

$$0 < x + y \to [0 < x \vee 0 < y]$$

We show that $D(a)$ is covered by $\{D(a) \mid Pos(a)\}$. Because

$$D(a) \vdash \bigvee_{r>0} D(a-r)$$

it is enough to show that, for a given $r > 0$, the open $D(a-r)$ is covered by $\{D(a) \mid Pos(a)\}$. But we have $||a^+|| < r$ or $0 < ||a^+||$. If $0 < ||a^+||$ we have $Pos(a)$ and $D(a-r)$ is covered by $D(a) = \{D(a) \mid Pos(a)\}$. If $||a^+|| < r$ then we have $D(a-r) = 0$ and hence $D(a-r)$ is covered by $\{D(a) \mid Pos(a)\}$. □

The converse of the theorem holds as well.

Theorem 2.2. *If $Max(R)$ is an open locale then all elements of R are normable.*

In contrast, the locale $Sp(R)$ don't seem to be open. It seems that if $Sp(R)$ is open then we have that $D(a)$ is positive iff $\neg(a \leq 0)$.

3. Building Points

Assume now that R is separable, that is we can find a sequence a_n of elements such that for any $a \in R$ and $r > 0$ there exists n such that $|a - a_n| \leq r$. Using dependent choice, we show that we build effectively a point in each open $D(a)$ such that $Pos(a)$. Let us write $a \in (p, q)$ the open $D(q-a) \wedge D(a-p)$. The idea is to choose $q_n \in \mathbb{Q}$ such that all open

$$D(a) \wedge a_1 \in (q_1 - 1/2, q_1 + 1/2) \wedge \cdots \wedge a_n \in (q_n - 2^{-n}, q_n + 2^{-n})$$

are positive. If $b \in R$ we can find a_{k_n} such that a_{k_n} converges to b. It can then be shown that q_{k_n} converges to a limit l. For this it is enough to notice that if the open

$$x \in (p, q) \wedge x \in (r, s)$$

is positive then (p, q) meets (r, s). If we take $\phi(b) = l$ we have defined a point ϕ which is in $D(a)$.

References

[1] Th. Coquand About Stone's Notion of Spectrum submitted

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