Inductive Definitions and ω-rule

So far, the only objects we have considered are natural numbers

It is direct (as was noticed by Gödel, Gentzen) to extend all the previous constructions to theory of *finite* objects, like list, trees, matrices, ...

We extend the notions of constructive objects by allowing countably branching well-founded trees

Borel "well-defined" sets

The first example of such "infinitary" objects is given by Borel sets (1898) that Borel called first "measurable" and then "well-defined" sets

Borel subsets of Cantor spaces have an inductive definitions

- a propositional formula is a Borel set
- if we have a sequence of Borel sets A_n then $\cap_n A_n$ and $\cup_n A_n$ are Borel sets

We see Borel sets as "symbols" (that are not syntactical objects however)

Countable ordinals

Countable ordinals are represented as well-founded trees

- 0 is an ordinal
- if we have a sequence of ordinals x_n then sup (x_n) is an ordinal

Intuitively sup (x_n) represents the supremum of all ordinals $x_n + 1$

σ-Boolean algebra

Boolean algebra B with an infinitary operation

$\vee_n x_n \in B$ for any sequence $x_n \in B$

If we have such a Boolean algebra we can interpret each closed formula A of PA as an element $[\![A]\!] \in B$

$$[\![\forall x.A(x)]\!] = \wedge_n [\![A(n)]\!] \qquad [\![\exists x.A(x)]\!] = \vee_n [\![A(n)]\!]$$

in such a way that if A is provable in PA then $[\![A]\!] = 1 \in B$

σ-Boolean algebra

If B is non trivial, we have another proof of the consistency of Peano arithmetic

We are going, following Gentzen, to construct the *free* σ-complete Boolean algebra, and show that it is non trivial

Propositional ω-logic

The *formulae* are described inductively

- a propositional formula is a formula
- If A_n is a sequence of formulae then so is $\wedge_n A_n$ and $\vee_n A_n$

We can define $\neg A$ by induction on A

A *sequent* $\Gamma, \Delta, \ldots$ is a finite set of formulae $A_1, \ldots, A_k$

Propositional ω-logic

We define $\vdash \Delta$ as an inductive definition

- $\vdash \Delta$ if Δ contains propositional formulae $A_1, \ldots, A_p$ such that $A_1 \vee \ldots \vee A_p$ is a tautology
- $\vdash \Delta, \wedge_n A_n$ if $\vdash \Delta, A_n$ for *all* n
- $\vdash \Delta$ if $\vee_n A_n \in \Delta$ and $\vdash \Delta, A_n$ for *some* n

A proof of $\vdash \Delta$ can be thought of as a well-founded tree

Lemma 1: If $\vdash \Delta$ and $\Delta \subseteq \Gamma$ then $\vdash \Gamma$

Lemma 2: If $\vdash \Delta, \wedge_n A_n$ then $\vdash \Delta, A_n$ for all n

Cut-elimination

Notice that the system does not state any cut-rule

Instead we show that the cut-rule is *admissible*

Theorem: If $\vdash \Delta, A$ and $\vdash \Gamma, \neg A$ then $\vdash \Delta, \Gamma$

The proof is by induction first on A and then on the proof of $\vdash \Delta, A$ and $\vdash \Gamma, \neg A$

We can then define $A \leq B$ iff $\vdash \neg A, B$

The transitivity of $\leq$ follows from admissibility of the cut-rule

The reflexivity of $\leq$ is $\vdash A, \neg A$ which is proved by induction on A

Cut-elimination

Theorem: The set of all formulae forms a σ-complete Boolean algebra, which is actually the *free* σ-complete Boolean algebra

Theorem: We have $A \leq B$ iff A seen as a Borel set is a subset of B seen as a Borel set

This is how Martin-Löf describes the Borel sets in his book "Notes on Constructive Mathematics"

Gentzen's first consistency proof

This first version was rejected by the referees (Weyl? and Bernays), for wrong reasons, it seems

Later (1970) Bernays presented this version

It contains in germ the idea of the ω-rule and suggest naturally a game interpretation

- $\vdash \Gamma$ if Γ contains a true atomic sentence
- $\vdash \Gamma, A_1 \wedge A_2$ if $\vdash \Gamma, A_1$ and $\vdash \Gamma, A_2$
- $\vdash \Gamma, \forall x.A(x)$ if $\vdash \Gamma, A(n)$ for all n
- $\vdash \Gamma$ if $A_1 \vee A_2 \in \Gamma$ and $\vdash \Gamma, A_i$ for $i = 1$ or $i = 2$
- $\vdash \Gamma$ if $\exists x.A(x) \in \Gamma$ and $\vdash \Gamma, A(n)$ for some n

Gentzen's first consistency proof

Notice the asymmetric treatment of $\forall$ and $\exists$

Any such proof can be interpreted in an "interactive" way, as a *game* between the proof and an opponent

At each move, the proof chooses one formula in the sequent, adding one instance of $\exists x.A$ if it chooses this formula

If it chooses a formula $A_1 \wedge A_2$ or $\forall x.A(x)$ the opponent plays by replacing this formula by an instance

The proof wins as soon as there is a true atomic formula

Proof tree = strategy

Branch of the tree = possible play

Gentzen's first consistency proof

A proof of $\vdash A$ can be thought of as the "finitary meaning" of the classical validity of A

For instance a proof of $\vdash \exists x \forall y. f(x) \leq f(y)$ explains what means the classical "truth" of $\exists x \forall y. f(x) \leq f(y)$

With this meaning, what is meant is not a natural number for x, but a strategy for finding eventually x where we are allowed to backtrack in our choice

Gentzen's first consistency proof

Example with $f(0) = 5, f(1) = 7, f(2) = 3, f(3) = 4, \ldots$

Proof $x = 0$

$$\vdash \exists x \forall y. f(x) \leq f(y), \underline{\forall y. f(0) \leq f(y)}$$

Opponent $y = 3$

$$\vdash \exists x \forall y. f(x) \leq f(y), f(0) \leq f(3)$$

Proof $x = 3$

$$\vdash \exists x \forall y. f(x) \leq f(y), f(0) \leq f(3), \underline{\forall y. f(3) \leq f(y)}$$

Opponent $y = 1$

$$\vdash \exists x \forall y. f(x) \leq f(y), f(0) \leq f(3), f(3) \leq f(1)$$

The proof wins!

Cut-elimination

We get another proof of consistency of arithmetic, since it is direct that there is *no* cut-free proof of $1 = 0$

Also, if we have a cut-free proof of a statement $\exists x.A(x)$ with A quantifier-free, then we can extract from this proof a witness n_0 such that $A(n_0)$ holds

But we get more than consistency: we explain the classical truth of arithmetical statements

Cut-elimination

With the usual first-order formulation of Peano arithmetic in sequent calculus we do not have a complete calculus with only cut-free proofs

With the ω-rule we get a complete cut-free system

A cut-free proof of a formula can be seen as a constructive explanation of the classical truth of this formula

It is the same for Borel subsets of Cantor space: a proof tree for $A \leq B$ can be seen as a constructive explanation of inclusion between Borel subsets, thought of as set of points

Generalised inductive definition

We have manipulated objects that can be thought of as well-founded countably branching tree, given by inductive definitions

The logic of such objects is called ID_1

The 1 refers to the fact that the branching is at most countable

If we use also classical logic when reasoning about such objects the logic is called ID_1^c

A concrete example in $\mathbf{ID}_1$

We represent the notion of well-quasi-ordering

First we define embedding $w \sqsubseteq w'$ between finite binary words, with ϵ empty word

$$\epsilon \sqsubseteq 0011 \qquad 011 \sqsubseteq 1000101 \qquad 11 \sqsubseteq 1010$$

A finite sequence of words $\sigma = w_0 \dots w_{n-1}$ is *good* iff $G(\sigma)$ iff there exists $i < j < n$ such that $w_i \sqsubseteq w_j$

A finite sequence of words $\sigma = w_0 \dots w_{n-1}$ is *barred by* G iff $B(\sigma)$ if $G(\sigma)$ *or* $B(\sigma w)$ for all w

Higman's lemma (particular case) states that $B(\sigma)$ holds for all σ

A concrete example in $\mathbf{ID}_1$

Notice that for this example, $G(\sigma)$ is a *decidable* property

$B(\sigma)$ will be also decidable, but is not decidable a priori

If we start from an arbitrary decidable property $G(\sigma)$ the corresponding predicate $B(\sigma)$ of being barred by G will *not* be decidable

The main problem in the next lecture will be to show, starting with an arbitrary decidable G, that we can assume $B(\sigma)$ to be decidable without having a contradiction

A concrete example in ID_2

We represent the tree of *minimal bad sequences*

We consider $w' < w$ lexicographic ordering on binary words

We define $M(\sigma)$ by induction on σ

$M(\epsilon)$ holds and $M(\sigma w)$ holds iff $M(\sigma)$ and $B(\sigma w')$ holds for all $w' < w$

This definition is in ID_1

We can then state that G is a bar on the tree defined by M (definition in ID_2)

$B_M(\sigma)$ iff $G(\sigma)$ or $B_M(\sigma w)$ for all w such that $M(\sigma w)$

The "minimal bad sequence" argument can then been thought of as a proof of $B_M(\epsilon) \rightarrow B(\epsilon)$

Generalised inductive definition

For arithmetic, or reasoning about finite objects, we have seen that classical logic can be explained in term of intuitionistic logic

Does this work for ID_1?? There is a problem because $B'(\sigma) = \neg\neg B(\sigma)$ does not satisfy a priori

$$(\forall w.B'(\sigma w)) \to B'(\sigma)$$

Indeed, one would need something like

$$(\forall w.\neg\neg A(w)) \to \neg\neg(\forall w.A(w))$$

which is not valid intuitionistically

In the next lecture, we shall give a reduction of ID_1^c to ID_1

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Exercice

In ω-logic gives a proof of, with $f : \mathbb{N} \rightarrow \mathbb{N}$

$$\vee_n.(f(n) \neq 0 \wedge f(n) \neq 1), \wedge_n \vee_{m>n} .f(m) = 0, \wedge_n \vee_{m>n} .f(m) = 1$$

which states that if f takes only the value 0 or 1 then it takes infinitely many times the value 0, or infinitely many time the value 1

Explain why the last statement is not intuitionistically valid