Krull Dimension of Distributive Lattices

Thierry Coquand and Henri Lombardi

Spectral spaces

Stone (1936)

"Topological Representation of Distributive Lattices and Brouwerian Logics"

Hochster (1969) "Prime ideal structure in commutative rings" Trans. Amer. Math. Soc. 142

exactly the spaces that are spectrum of a commutative ring

As point-free spaces: distributive lattices

Write $Sp(D)$ the space defined by D

Logical description

A distributive lattice can be thought of as the Tarski-Lindenbaum algebra of a propositional coherent theory

- $D(1), \qquad \neg D(0)$
- $D(a+b) \to D(a) \vee D(b)$
- $D(ab) \to D(a)$
- $D(a) \wedge D(b) \to D(ab)$

The *points* of the spectral space corresponds to the *models* of this theory

Scott's entailment relations

Proof theory, cut-elimination

Entailment relations, like clauses in resolution calculus

Examples: spectrum of ring

$$D(a_1), \ldots, D(a_n) \vdash D(b_1), \ldots, D(b_m)$$

iff

the monoid generated by $a_1, \ldots, a_n$ meets the ideal generated by $b_1, \ldots, b_m$

The models (points) are complement of prime ideals

Formal version of Hilbert's Nullstellensatz

Spectrum of a commutative group

If A commutative group

- $D(a+b) \to D(a) \vee D(b)$
- $\neg(D(a) \wedge D(-a))$

$$D(a_1), \ldots, D(a_n) \vdash D(b_1), \ldots, D(b_m)$$

iff

A non empty sum of the a_i is equal to a sum of the b_j

Spectrum of l-group

If A is an l-group with unit

- $\neg D(a)$ if $a \leq 0$
- $D(a \vee b) \to D(a) \vee D(b)$
- $D(a + b) \to D(a) \vee D(b)$
- $\neg(D(a) \wedge D(-a))$
- $D(1)$

$$D(a_1), \ldots, D(a_n) \vdash D(b_1), \ldots, D(b_m)$$

iff for some N

$$a_1^+ \wedge \cdots \wedge a_n^+ \leq N(b_1^+ \vee \cdots \vee b_m^+)$$

How to get compact Hausdorff spaces?

Non Hausdorff spaces; the models (points) are ordered

Look at the maximal points

Normal spectral spaces, if $x \vee y = 1$ we can find a, b

$$ab = 0 \qquad x \vee a = 1 \qquad y \vee b = 1$$

Theorem: D is normal iff any point is contained in a unique maximal point. The topology on maximal points is compact Hausdorff.

Example of the spectrum of a l-group: we get a space X and the elements of the l-group define elements of $C(X)$

Finite case: Birkhoff duality

Duality between discrete *finite lattices* and discrete finite posets

The *models* correspond to *irreducible elements*

In this case: Krull dimension is

length of the longest chain

General case

Theory T_n associated to a lattice D

atomic propositions $D_i(a)$, $i \leq n$

- $\vdash\ D_i(1)$
- $D_i(0)\ \vdash$
- $D_i(a \vee b)\ \vdash\ D_i(a), D_i(b)$
- $D_i(ab)\ \vdash\ D_i(a)$
- $D_i(a), D_i(b)\ \vdash\ D_i(ab)$
- $D_{i+1}(a) \vdash D_i(a)$

Universal solution to $D_i : D \to L$ with

$$D_n \leq \cdots \leq D_0$$

We write $Kr_n(D)$ the solution

Proof-theoretical analysis

Main theorem: we have

$$D_0(u_0), \ldots, D_n(u_n) \vdash D_0(j_0), \ldots, D_n(j_n)$$

iff there exists $x_1, \ldots, x_n$ such that

$$x_1 \wedge u_0 \leq j_0$$

$$x_2 \wedge u_1 \leq j_1 \vee x_1$$

$$\ldots$$

$$u_n \leq j_n \vee x_n$$

Krull dimension, constructively

For instance

$$D_0(u) \vdash D_1(u)$$

iff there exists x

$$x \wedge u \leq 0$$

$$1 \leq u \vee x$$

iff u has a complement

Theorem: D has Krull dimension $\leq n-1$ iff for any $x_1, \ldots x_n$

$$D_0(x_1), \ldots, D_{n-1}(x_n) \vdash D_1(x_1), \ldots, D_n(x_n)$$

Case of Heyting algebra

Implicative lattice (Skolem) or Heyting algebra

$$a \wedge b \leq c \Longleftrightarrow a \leq b \to c$$

If L is an implicative lattice, we have

$$D_0(u_0), \ldots, D_n(u_n) \vdash_S D_0(j_0), \ldots, D_n(j_n)$$

iff

$$1 = u_n \to (j_n \vee (u_{n-1} \to (j_{n-1} \vee \ldots (u_0 \to j_0))))$$

Case of Heyting algebra

dimension 0

$$1 = x \vee \neg x$$

dimension ≤ 1

$$1 = x_2 \vee (x_2 \to (x_1 \vee \neg x_1))$$

and so on

An implicative lattice L is of dimension $\leq n-1$ iff

$$1 = x_n \vee (x_n \to \ldots (x_2 \vee (x_2 \to (x_1 \vee \neg x_1))) \ldots)$$

Example: finite discrete lattices

Krull dimension of spectral spaces

X is is of dimension $\leq n-1$ iff

$$1 = x_n \cup (x_n \to \dots (x_2 \cup (x_2 \to (x_1 \cup \neg x_1))) \dots)$$

for any compact open $x_1, \dots, x_n$

where $U \to V$ is the greatest open W such that $U \cap W \subseteq V$

When is T_n decidable?

$Kr_n(D)$ discrete iff T_n decidable

(Joyal) We can have $Kr_1(D)$ non discrete even if D is discrete

Generic discrete lattice

Wraith's example: with topos of "Sets-in-Time"

When is T_n decidable?

Theorem: If lattice D is a discrete implicative lattice then each $Kr_n(D)$ is discrete.

Theorem: If R is a coherent noetherian ring then $Zar(R)$ is a discrete implicative lattice

Particular case: $R = K[x_1, \ldots, x_n]$

Corollary: If R is coherent, noetherian and strongly discrete then each lattice $Kr_n(R)$ is discrete.

Covering dimension: normal spectral spaces

D normal defines a compact Hausdorff space X

Covering dimension: introduced by Lebesgue

Proposition: (Carral-Coste) the covering dimension of $Sp(D)$ is equal to the one of X

Theorem: (Carral-Coste) the covering dimension of D is $\leq$ the Krull dimension of D

Interpretation between theories

proper maps $f : Sp(L) \rightarrow Sp(D)$ corresponds to morphisms $f^* : D \rightarrow L$

f^* can be thought of as an interpretation between two theories

Proposition: f is surjective iff

$$f^*(a) \leq f^*(c) \quad iff \qquad a \leq c$$

iff the interpretation f^* is conservative

Going-down theorem

$f : Sp(L) \to Sp(D)$ going-down property

If $f(\beta) = \alpha$ and $\alpha_1 \leq \alpha$ we can find $\beta_1 \leq \beta$ such that $f(\beta_1) = \alpha_1$

Theorem: f has the going-up property iff whenever $f^*(a) \wedge y \leq f^*(c)$ we can find x such that $a \wedge x \leq c$ and $y \leq f^*(x)$

This means

$$f^*(a \to c) = f^*(a) \to f^*(c)$$

in an implicative lattice

Going-up theorem

$f : Sp(L) \to Sp(D)$ going-up property

If $f(\beta) = \alpha$ and $\alpha \leq \alpha_1$ we can find $\beta \leq \beta_1$ such that $f(\beta_1) = \alpha_1$

Theorem: f has the going-up property iff whenever $f^*(a) \leq y \vee f^*(c)$ we can find x such that $a \leq x \vee c$ and $f^*(x) \leq y$

This means

$$f^*(a - c) = f^*(a) - f^*(c)$$

in a substractive lattice

Going-up and going-down in algebra

We get constructive proofs of the two basic results

Theorem: If $i : R \to S$ and S integral over R then $i : Sp(S) \to Sp(R)$ has the going-up property

Theorem: If $i : R \to S$ and S integral over R and R is integrally closed then $i : Sp(S) \to Sp(R)$ has the going-down property