NOVIKOV'S CUT-ELIMINATION

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Introduction

Following Novikov, we define the notion of *regular* set. It will be convenient (but not essential) to use non well-founded sets. This illustrates also that cut-elimination is *not* proved by induction on the cut-formula like in other proofs (Gentzen, Tait).

If we apply this to hereditarily finite sets (over a given set of urelements), we get a cut-elimination for ordinary propositional logic.

1. Formulae

We suppose given a set of pair of distinct elements u_0, u_1 such that

$$u_0 = \{u_1\}, u_1 = \{u_0\}$$

We shall interpret intuitively each set as a propositional formula, with $\{x_i \mid i \in I\}$ meaning intuitively

$$\neg \bigwedge_i x_i$$

(if I undersand it right, this is the structure described in Wittgenstein's Tractacus)

Thus, we have intuitively $u_0 = \neg u_1, u_1 = \neg u_0$.

2. Regular formulae

Regular set shall correspond to true propositions.

We say that a is *regular* iff

1. we can find $u_0, u_1 \in a$ or
2. there exists $x \in a$ such that for all $y \in x$, the set $a \cup y$ is regular.

Thus, the empty set $\emptyset$ is not regular, and if $\emptyset \in a$ then a is regular.

If a is well-founded, it is the case classically that a is regular or $\{a\}$ is regular.

Lemma 2.1. *if $a \subseteq b$ and a is regular then so is b.*

Proof. This is clear by induction on the proof that a is regular. □

We say that a is *eliminable* iff

$$\forall b.[\forall x \in a.b \cup x \ regular] \to b \ regular$$

Lemma 2.2. *If we have*

1. $\forall y \in x.a \cup y$ *eliminable*

2. $b \cup x$ *regular*
3. $\forall z \in a. b \cup z$ *regular*

then b *is regular.*

Proof. By induction on the proof that $b \cup x$ is regular. There are three cases

1. $u_0, u_1 \in b \cup x$
2. for some $t \in b$, $\forall x \in t. b \cup x \cup v$ is regular
3. for some $t \in x$, $\forall x \in t. b \cup x \cup v$ is regular

The first case splits in four subcases.

If $u_0, u_1 \in b$ it is direct that b is regular.

If $u_0 \in b, u_1 \in x$ then $a \cup u_1 = a \cup \{u_0\}$ is eliminable by (1) and $b \cup u_0 = b \cup \{u_1\}$ is regular since $u_0, u_1 \in b \cup \{u_1\}$. Hence by 3, b is regular.

If $u_1 \in b, u_0 \in x$ this is the same as previous case.

If $u_0, u_1 \in x$ then $a \cup u_1 = a \cup \{u_0\}$ and $a \cup u_0 = a \cup \{u_1\}$ are eliminable by 1. By 3 and lemma 1, each $b \cup \{u_0\} \cup z$ is regular for $z \in a$, and also for $z = u_0$, since then $u_0, u_1 \in b \cup \{u_0\} \cup z$. Hence since $a \cup \{u_0\}$ is eliminable $b \cup \{u_0\}$ is regular. But then each $b \cup z$ is regular for $z \in a \cup \{u_1\}$, and hence b is regular since $a \cup \{u_1\}$ is eliminable.

This ends the analysis of the first main case.

The second case is direct: by lemma 1, for each $v \in t$ we have $\forall z \in a. b \cup z \cup v$ is regular. Hence by induction each $b \cup v$ is regular, and hence b is regular.

The third case is the most interesting: first like in the second case, by lemma 1, for each $v \in t$ we have $\forall z \in a. b \cup z \cup v$ is regular. Hence by induction each $b \cup v$ is regular. Since by 1 $a \cup t$ is eliminable, and by 3 $b \cup z$ is regular also for each $z \in a$, we get that b is regular. □

The main result is Novikov's cut-elimination result.

Theorem 2.3. *If* a *is regular and for all* $x \in a$, $b \cup x$ *is regular, then* b *is regular.*

Proof. We show that if a is regular it is eliminable by induction on the proof that a is regular. This follows from lemma 2, which shows that

$$(\forall y \in x)[a \cup y \textit{ eliminable}] \rightarrow a \cup \{x\} \textit{ eliminable}$$

and hence if $x \in a$

$$(\forall y \in x)[a \cup y \textit{ eliminable}] \rightarrow a \textit{ eliminable}$$

It is also clear that if $b \cup u_0 = b \cup \{u_1\}$ and $b \cup u_1 = b \cup \{u_0\}$ are regular, so is b, by induction on the proofs that $b \cup u_0$ and $b \cup u_1$ are regular. Notice that we have to use $u_0 \neq u_1$ in the base case. □

Corollary 2.4. *If* $a \cup b$ *and* $\{a\}$ *are regular then so is* b.

Corollary 2.5. *There is no set* a *such that both* a *and* $\{a\}$ *are regular.*

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