Stack models of type theory

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Stack models of type theory

The goal is to refine the notion of *sheaf model*, which is defined for *simple* type theory, to *dependent* type theory

A sheaf is defined by a gluing condition of *compatible* local data

(By unique choice this also defines a structure)

The notion of *compatibility* refers to the notion of *identification*, so it is natural that the univalence axiom, and the stratification of the notion of identification play a crucial role

Stack models of type theory

One application: to show that the principles

$(\Pi(n:N)\ \|A(n)\|) \rightarrow \|\Pi(n:N)A(n)\|$ (countable choice)

$(\Pi(n:N)\ \|B+T(n)\|) \rightarrow \|\Pi(n:N)(B+T(n))\|$

($T(n)$ decidable subsingleton)

are independent of type theory with univalence

Another potential application is to design a $\langle\!\langle$reactive type theory$\rangle\!\rangle$ extending functional reactive programming

Groupoid model

We first try to use the groupoid model (model of one univalent universe)

It is also a model of propositional truncation

$\|A\|$ same objects of A but exactly *one* path between two objects

Do we have a counter-model of

$\Pi(A : N \to U)\ (\Pi(n : N)\ \|A\ n\|) \to \|\Pi(n : N)(A\ n)\|$

if countable choice does not hold in the meta-theory?

U is the groupoid of sets with isomorphisms

Groupoid model

For each given A, classically we can prove $\|A\| \to A$

However, even classically, $\Pi(A : U)\ \|A\| \to A$ is empty

One surprise(?) is that, with this interpretation, countable choice *always* holds

$\Pi(A : N \to U)\ (\Pi(n : N)\ \|A\ n\|) \to \|\Pi(n : N)(A\ n)\|$

We define an operation $c\ A\ f = f$ and on path $c\ \alpha\ \omega = 0$

Groupoid model

This means that, in order to get an independence proof of countable choice, we cannot use the following approach: develop the groupoid model in a setting where we have universes and countable choice does not hold (e.g. suitable sheaf model of CZF)

Sheaf model of universe

Let $\mathcal{U}$ be a Grothendieck universe

We suppose given a topological space with basic open sets $U, V, W, \dots$

We can define $F(V)$ to be the collection of all $\mathcal{U}$-presheaves on V

There is a natural restriction operation $F(V) \to F(W)$ if $W \subseteq V$

So we get a presheaf

If we instead take $F(V)$ to be the collection of all $\mathcal{U}$-sheaves on V

There is a natural restriction operation $F(V) \to F(W)$ if $W \subseteq V$

Gluing local data is possible, but only *up to isomorphism*

Sheaf model of universe

I learnt this problem from Martin Escardó and Chuangjie Xu

A related question is discussed in EGA 1, 3.3.1

We replace strict equality by 《path》 equality (isomorphism)

How to glue compatible (in the sense of isomorphism) locally defined sheaves?

When doing this, compatibility 2 by 2 is not enough: we should have compatibility 3 by 3 with the cocycle condition

Stack models

Instead we use the notion of *stack* (j.w.w. Bassel Mannaa and Fabian Ruch)

This sounds natural but one could expect coherence problems

The original insight that this might actually work is due to Bassel Mannaa

We have a family of groupoids $\Gamma(U)$ for U basic open with restriction maps that are now (strict) *groupoid maps*, for $V \subseteq U$

$\Gamma(U) \rightarrow \Gamma(V)$

$a \longmapsto a|V$

Descent data and stack structure

If $C = (U_i)$ is a covering of U the gluing structure is formulated as follows

We write $U_{ij} = U_i \cap U_j$ if U_i meets U_j

We first define the groupoid $\Gamma(C)$ of *descent data*

A descent data is a family a_i in $\Gamma(U_i)$ with paths $\omega_{ij} : a_i \to a_j$ in $\Gamma(U_{ij})$ satisfying the cocycle condition

Descent data and stack structure

The descent data form a groupoid: a path $(a_i, \omega_{ij}) \to (b_i, \delta_{ij})$ is given by a collection of paths $a_i \to b_i$ such that the following diagram commutes in $\Gamma(U_{ij})$

$$\begin{tikzcd}
a_i \arrow[r, "\omega_{ij}"] \arrow[d] & a_j \arrow[d] \\
b_i \arrow[r, "\delta_{ij}"] & b_j
\end{tikzcd}$$

Descent data and stack structure

Any element a in $\Gamma(U)$ defines a 《constant》 descent data $a_i = a$ in $\Gamma(U_i)$ with $\omega_{ij} = 1$ in $\Gamma(U_{ij})$

We thus have a canonical map $\Gamma(U) \rightarrow \Gamma(C)$

Definition: Γ *is a* stack *if this map is an* equivalence

Descent data and stack structure

This is one definition we can find in the literature

In order to interpret type theory, we need to *refine* this definition as follows

Descent data and stack structure

First, we ask for an explicit adjoint map $\Gamma(C) \to \Gamma(U)$

This means that we have an *explicit* operation $\mathsf{glue}(a_i, \omega_{ij}) = a$ and an *explicit* operation which build paths $\alpha_i : a \to a_i$ such that $\alpha_i \cdot \omega_{ij} = \alpha_j$

In general we cannot hope to have the *strict* equality $a = a_i$ on U_i

The elements a and a_i are only *path* equal

The notion of stack is a *structure* (and not a simple *property*)

Descent data and stack structure

Second, if $V \subseteq U$ then we can consider the covering $C \cap V = (U_i \cap V)$ of V

We require a *strictly* commuting diagram (also one for the universal map)

$$\begin{tikzcd}
\Gamma(C) \arrow[r, "\text{glue}"] \arrow[d] & \Gamma(U) \arrow[d] \\
\Gamma(C \cap V) \arrow[r, "\text{glue}"] & \Gamma(V)
\end{tikzcd}$$

Descent data and stack structure

In particular $\mathsf{glue}(a_i, \omega_{ij})|V = \mathsf{glue}(a_i|V \cap U_i, \omega_{ij}|V \cap U_{ij})$

The gluing of compatible local data has to be ⟪uniform⟫ w.r.t. restriction

We can then define the notion of family of stacks and build a model of type theory with dependent products, sums, path types and one (univalent) universe

We stress the fact that we build a model of category with families: all required equations hold *strictly*

This is never discussed in the literature (which does not look at the question of interpreting dependent type theory): with the usual definitions, it is not even clear if stacks form a cartesian closed category

Universe

We give $C = (U_i)$ is a covering of U and F_i is a sheaf on U_i and we have isomorphisms $\varphi_{ij} : F_i|U_{ij} \to F_j|U_{ij}$ satisfying the cocycle condition

There is a canonical way to define a sheaf F on U

An element of $F(V)$ is a family a_i in $F_i(V \cap U_i)$ such that $\varphi_{ij}(a_i) = a_j$

We can check that F is a sheaf on U and we have isomorphims $F|U_i \to F_i$

We get a (uniform) stack structure

Gluing would not be uniform if defined using global choice as in EGA 1

Propositional truncation

If Γ is a stack, we define $\|\Gamma\|(U)$ as follows

Given by a set of objects, and there is exactly one path between two objects

The objects are defined inductively (well-founded trees)

-any object of $\Gamma(U)$ defines an object of $\|\Gamma\|(U)$

-if we have a covering $C = (U_i)$ of U and a family a_i of element in $\|\Gamma\|(U_i)$ this defines an element (a_i) of $\|\Gamma\|(U)$

We get in this way a (uniform) stack structure

Countable choice

The simplest counter model seems to be given by the lattice of basic open

$X_n = [0, 1/2^n) \qquad R_n = (0, 1/2^n) \subseteq X_n$

with X_n covered by X_{n+1} and R_n and $R_{n+1} = X_{n+1} \cap R_n$

We define $\varphi_0(n)$ to be X_n and $\varphi_1(n)$ to be R_0

$\varphi_0(n) \vee \varphi_1(n)$ is the total space $X_n \cup R_0 = X_0$

But both $\varphi_0(n)$ and $\varphi_1(n)$ are false at level X_l if $l < n$

So $\Pi(n : N)(\varphi_0(n) + \varphi_1(n))$ is empty at each level X_l

One point space

In a groupoid we can ⟪duplicate⟫ informations: we can consider a family of objects a_i with $a_i \to a_j$ satisfying the cocycle condition

Then we have an explicit ⟪choice⟫ operation which selects an object a and paths $a \to a_j$

E.g. in the groupoid of sets we have a family of sets A_i and isomorphisms $A_i \to A_j$ satisfying the cocyle condition

The ⟪canonical⟫ choice of gluing for this family is the limit of this diagram

Any definable groupoid has such an extra choice structure

We get a *new* model of type theory in this way

Inductive types

There is something subtle going on for interpreting the type of natural numbers

We interpret it by the constant presheaf $N(U) = N$

This is a sheaf *only* because the space is connected

This would not work for a disjoint covering, e.g. Cantor space

In the case of Cantor space a natural number at level U is given by a partition of U and a selection of natural numbers for each block of the partition

Inductive types

For a disjoint covering $U_1, \ldots, U_n$ we have to require an extra condition on the gluing operation, the *strict* equality

$a_i = \mathsf{glue}(a_1, \ldots, a_n)|U_i$

These issues can only be seen because we try to interpret type theoretic elimination rules with judgemental equalities, that are interpreted by strict equalities in the model

In general, we don't have $\mathsf{glue}(a|U_1, \ldots, a|U_n) = a$ (e.g. for the universe)

Inductive types

So we need two *different* kind of gluing operations for *connected* coverings and for *disjoint* coverings

E.g. for the principle, with $T\ n$ decidable subsingleton

$(\Pi(n:N)\ \|B+T\ n\|) \rightarrow \|\Pi(n:N)(B+T\ n)\|$

Andrew Swan considered the space $(0,1)\times C$ where C is Cantor space

The groupoids are family $\Gamma(U|b)$ where U basic open of $(0,1)$ and b basic open of Cantor space and we have two kind of coverings

$U|b$ covered by $U_0|b$ and $U_1|b$ (connected)

$U|b$ covered by $U|b_1,\dots,U|b_n$ (disjoint)

Sites

So far we have only looked at stacks over topological spaces

We can also consider topology defined by sites, e.g. Schanuel topos

What is interesting in this situation is that we have a new situation of 《self intersection》: even in the case of a covering with only *one* map, the cocycle condition is non trivial

Cubical stacks

All that we have done so far can be generalized to the notion of *cubical stacks* (thanks to several discussions with Christian Sattler)

What is crucial here is that we can consider new《path》models of type theory, e.g. a model where a type is interpreted by two types A and B and a *path* connecting A and B

Theorem: *This forms a model of cubical type theory, and hence of type theory with univalence and propositional truncation*

Follows from the fact that this can be seen as a model over the context $i : \mathbb{I}$

Presheafs

A presheaf Γ is given by a collection of sets

$\Gamma(I|U)$

where I finite set of names and U basic open

We have restriction maps

$\Gamma(I|U) \rightarrow \Gamma(J|U) \qquad \rho \longmapsto \rho f$

for $f : J \rightarrow I$ with the laws $\rho 1 = \rho$ and $\rho(fg) = (\rho f)g$ and

$\Gamma(I|U) \rightarrow \Gamma(I|V) \qquad \rho \longmapsto \rho|V$

if $V \subseteq U$ with the laws $\rho|U = \rho$ and $\rho|W = (\rho|V)|W$

Dependent presheaf

Given a presheaf Γ, a dependent presheaf $\Gamma \vdash A$ is given by a presheaf on the category of elements of Γ

Explicitely it is given by a family of sets $A(I|U,\rho)$ with ρ in $\Gamma(I|U)$ with restriction maps, for $f : J \to I$

$$u \mapsto uf \qquad A(I|U,\rho) \to A(J|U,\rho f)$$

and for $V \subseteq U$

$$u \mapsto u|V \qquad A(I|U,\rho) \to A(V,\rho)$$

Presheaf

We introduce the notation

$\vdash^I_V A$

to mean that A is a dependent type on the presheaf represented by $I|V$

Explicitely, it is given by a family of sets $A(f,W)$ for $f : J \rightarrow I$ and $W \subseteq V$ and restriction maps

All operations we consider will commute with substitutions and restrictions

If $\vdash^I_V \Gamma$ we define $\Gamma \vdash^I_V A$ to mean that A is a presehaf on the category of elements of Γ

Descent data

Given a covering $C = (V_0, V_1)$ of a basic open V with a non empty intersection $V_{01} = V_0 \cap V_1$ and $\vdash^I_V A$ we define the type of *descent data* $\vdash^I_V D_C(A)$

Descent data

The introduction rule for descent data is

$$\frac{\vdash^I_{V_0} a_0 : A \qquad \vdash^I_{V_1} a_1 : A \qquad \vdash^I_{V_{01}} a_{01} : \mathsf{Path}\ A\ a_0\ a_1}{\vdash^I_V (a_0, a_1, a_{01}) : D_C(A)}$$

Theorem: *If A has a composition structure then so has the type $D_C(A)$*

There is a canonical map $\vdash^I_V \lambda(a : A)(a, a, \langle i \rangle a) : A \to D_C(A)$

Stack structure

A *stack structure* is an *equivalence structure* for this map $A \to D_C(A)$

Stack structure

Let us write $\overline{a}$ for $(a, a, \langle i \rangle a)$ (we may write simply a if there is no possible ambiguity)

One way to express the stack structure is by giving two explicit operations

$\vdash^I_V \mathsf{ext}([\psi \mapsto a], d) : A$

given $a : A$ such that $\vdash^{I,\psi}_V \overline{a} = d : D_C(A)$, which restricts to a on ψ, and

$\vdash^I_V \tilde{\mathsf{ext}}([\psi \mapsto a], d) : \mathsf{Path}\ D_C(A)\ \mathsf{ext}([\psi \mapsto a], d)\ d$

which restricts to the constant path $\overline{a}$ on ψ

Stack structure

We can then define the stack structure by induction on the type

For instance the stack structure on $T = \Pi(x : A)B$ is defined by the equation (we give here only the definition for ext)

$$\mathsf{ext}([\psi \mapsto w], (w_0, w_1, w_{01}))\ a = \mathsf{ext}([\psi \mapsto w\ a], (w_0\ a, w_1\ a, \langle i \rangle w_{01}\ i\ a))$$

Stack structure for the universe

We only give the definition of $\mathsf{ext}([\psi \mapsto A], D) : U$ given $D = (A_0, A_1, A_{01})$ such that $A = D$ on ψ

We consider the type B of elements a_0, a_1, a_{01} with $a_0 : A_0$ on V_0 and $a_1 : A_1$ on V_1 and $a_{01} : \mathsf{Path}^i\ (A_{01}\ i)\ a_0\ a_1$ on V_{01}

B has a composition structure

Since $A = D$ on ψ we can consider the map $c_D : A \to B$ defined on ψ which sends a to $(a, a, \langle i \rangle a)$

Since A is a stack, this map is an *equivalence*

We can then define $\mathsf{ext}([\psi \mapsto A], D) = \mathsf{Glue}\ [\psi \mapsto (A, c_D)]\ B$ on $I|V$