The strongest principles of reasoning that were stated explicitly in the traditional intuitionistic literature (see Brouwer 1918 and, for a more explicit formulation, Brouwer 1924) are the principles of definition and proof by <u>transfinite</u> or <u>generalized induction</u> as formalized in the systems of Kleene and Vesley 1965 and Kreisel and Troelstra 1970. They are the principles that are needed in order to develop, for example, the theory of Baire space, Borel sets and ordinals of the second number class. It is true that Brouwer 1918 introduced the notion of species and even species of finite type but, since he did not touch upon the crucial question whether or not to allow species to be defined impredicatively, it is not possible to determine the proof theoretic strength that results from the introduction of the notion of species. In actual practice no more abstract way of defining species than by generalized induction seems to have been used.

Although the strength of the principles just mentioned suffices for large parts of mathematical practice, there are number theoretic problems arising in proof theory whose logical form is so simple that their meaningfulness can hardly be doubted but whose solution is known to require principles that are far more abstract than the principle of generalized induction. A prime example is the problem of consistency and normalization for second order arithmetic with the full impredicative comprehension axiom. To solve this problem, even so called <u>iterated generalized induction</u>, which is the strongest principle contemplated by Tait 1967, does not suffice.

This lack of proof theoretic strength has been a most serious objection against intuitionism as a basis for the whole of mathematics and also a stumbling block for constructive proof theory.

The situation was radically changed when Kreisel 1967 pointed out the intuitionistic significance of the impredicative theory of species, that is, second order arithmetic with intuitionistic instead of classical logic. Once the theory of species is accepted, one gets an intuitionistic consistency proof for classical analysis by extending in the obvious way the double negation interpretation of Kolmogorov 1925, Gödel 1933 and Gentzen 1933. And, more informatively, one gets an intuitionistic proof of normalization for intuitionistic second order logic (see Girard 1970, Martin-Löf 1970 and Prawitz 1970) improving Prawitz's 1968 earlier completeness theorem for the cut free rules. The proof theoretical analysis adds to the conviction that there be no conflict between the impredicativity and the intuitionistic interpretation of the notion of function and the logical operations. For example, it yields a mechanical method of transforming an arbitrary numeral m and a formal proof of $\forall x \exists y A(x,y)$ into a numeral n and a formal proof of A(m,n).

Having accepted the theory of species, there is no reason not to go one step further and accept the full intuitionistic theory of finite types, that is, the simple theory of finite types with intuitionistic instead of classical logic, with the Peano axioms but without the axiom of extensionality. This theory is proof theoretically as strong as the corresponding classical theory with extensionality, that is, as strong as Principia Mathematica. To see this one interprets, first, the classical theory with extensionality into the classical theory without extensionality by means of Takeuti 1953 or Gandy 1956 and, second, the classical theory without extensionality into the corresponding intuitionistic theory by means of the double negation interpretation.

The simple theory of finite types, although proof theoretically quite strong, has some unnatural limitations (for example, it permits only finite iterations of the power operation) and, above all, it is not adequate for a formalization of those parts of mathematics that talk about arbitrary sets and not just sets of natural numbers, sets of sets of natural numbers, and so on. Therefore, the idea has occurred to me of formulating a <u>general intuitionistic theory of types</u> which could serve as a logical basis for intuitionistic mathematics in somewhat the same way as set theory has served as a basis for classical mathematics. The rest of my talk will be devoted to an informal description of this theory. For a formal account, see Martin-Löf 1971. As for the strength of the theory, the best that I can say at present is that it exceeds that of Zermelo's set theory.

We shall think of <u>mathematical objects</u> or <u>constructions</u>. Every mathematical object is of a certain kind or type. Better, a mathematical object is always given together with its type, that is, it is not just an object, it is an object of a certain type. I shall use the notation

$$x \in A$$

to express that x is an object of type A. A type is well-defined if we understand (or grasp as Kreisel would say) what it means to be an object of that type. The types are themselves mathematical objects, namely, those objects whose type is the type of types. I shall denote the type of types by the symbol V. Note that V is itself a type, namely, the type of types, and hence an object of type V. In symbols,

$$V \in V.$$

The type of types introduces a strong kind of selfreference which, as pointed out by Gödel 1964, transcends the cumulative hierarchy notion of set and may seem to verge on the paradoxes, but which is actually being used in category theory, notably, in the construction of the category of all categories.

A proposition will be represented by a certain type, namely, the type of proofs of that proposition. Conversely, if we accept the abstract intuitionistic explanation of the notion of proposition, according to which a proposition is defined by prescribing how we are allowed to prove it, then we may think of a type as a proposition, namely, the proposition which we prove by exhibiting an object of that type. In other words, we may simply identify propositions and types and interpret

$$x \in A$$

alternatively as

x is a proof of the proposition A.

On the formal level, the analogy between propositions and types was discovered by Curry and Feys 1958.

The idea of the type of types is forced upon us by accepting simultaneously each of the following three principles. First, quantification over propositions as in impredicative second order logic. Second, Russell's doctrine of types according to which the ranges of significance of propositional functions form types so that, in particular, it is only meaningful to quantify over all objects of a certain type. Third, the identification of propositions and types. Suppose namely that quantification over propositions is meaningful. Then, by the doctrine of types, the propositions must form a type. But, if propositions and types are identified, then this type is at the same time the type of types.

Suppose now that we have defined a <u>function</u>, <u>rule</u> or <u>method</u> which to an arbitrary object x of type A assigns a type B(x). Then the <u>cartesian product</u>

$$(\Pi x \in A)B(x)$$

is a type, namely, the type of functions which take an arbitrary object x of type A into an object of type B(x). Clearly, we may <u>apply</u> an object f of type $(\Pi x \in A)B(x)$ to an object x of type A, thereby getting an object

$$f(x)$$

of type B(x). If we think of B(x) as a proposition rather than a type, $(\Pi x \in A)B(x)$ is the <u>logical product</u> or conjunction of the propositions B(x) obtained by letting x range over A. A proof of $(\Pi x \in A)B(x)$ is a function which to an arbitrary object x of type A assigns a proof of B(x). Functions may be introduced by <u>explicit definition</u>. That is, if we build up a term from constants for already defined objects and a variable x that denotes an arbitrary object of type A and if this term t denotes an object of type B(x), then we may introduce a function f of type $(\Pi x \in A)B(x)$ by means of the schema

$$f(x) = t.$$

Here and in the following all mention of parameters is suppressed.

If B(x) is defined to be one and the same type B for every object x of type A, then $(\Pi x \in A)B(x)$ will be abbreviated

$$A \rightarrow B.$$

It is the type of functions from A to B. Thinking of A and B as propositions, it is the proposition

A implies B.

The *power type* of a type A

$$P(A) = A \rightarrow V$$

is the type of propositional functions defined on A or, in intuitionistic terminology, the type of *species* of objects of type A.

If A and B are types, so is the *disjoint union*

$$A + B$$

which is the type of objects of the form $i(x)$ with x of type A or $j(y)$ with y of type B. Here i and j denote the canonical injections. In case A and B are thought of as propositions, A + B is their disjunction. Suppose that g and h are functions of type $(\Pi x \in A)C(i(x))$ and $(\Pi y \in B)C(j(y))$, respectively. Then we may define a function f of type $(\Pi z \in A + B)C(z)$ by means of the schema

$$\begin{cases} f(i(x)) = g(x), \\ f(j(y)) = h(y). \end{cases}$$

Given a function which to an object x of type A assigns a type $B(x)$, we may form the *disjoint union*

$$(\Sigma x \in A)B(x)$$

which is the type of pairs (x,y) where x and y are objects of type A and $B(x)$, respectively. When we think of $B(x)$ as a proposition rather than a type, $(\Sigma x \in A)B(x)$ is the proposition

there <u>exists</u> an object x of type A such that B(x)

which we prove by exhibiting a pair (x,y) where x is an object of type A and y is a proof of B(x). A third interpretation of $(\Sigma x \in A)B(x)$ is as

the type of all objects x of type A <u>such that</u> B(x)

because, from the intuitionistic point of view, to give an object x of type A such that B(x) is to give x <u>together</u> with a proof y of the proposition B(x). This interpretation of the notion of such that is implicitly used by Bishop 1967. However, its explicit formulation requires us to consider proofs as mathematical objects. (See also Kreisel 1967 for a discussion of this point.) Given a function g of type $(\Pi x \in A)(\Pi y \in B(x))C((x,y))$ we may introduce a function f of type $(\Pi z \in (\Sigma x \in A)B(x))C(z)$ by means of the schema

$$f((x,y)) = g(x,y).$$

For example, the left and right projections p and q of type $(\Sigma x \in A)B(x) \rightarrow A$ and $(\Pi z \in (\Sigma x \in A)B(x))B(p(z))$, respectively, are defined by putting

$$\begin{cases} p((x,y)) = x, \\ q((x,y)) = y. \end{cases}$$

In the special case when B(x) is defined to be one and the same type B for every object x of type A,

$(\Sigma x \in A)B(x)$ is abbreviated

$$A \times B.$$

It is at the same time the cartesian product and the conjunction of A and B.

For a nonnegative integer n we introduce a type N_n with precisely the n objects 1, 2, ..., n. Given objects $c_1, \ldots, c_n$ of types C(1), ..., C(n), respectively, we may then define a function f of type $(\Pi x \in N_n)C(x)$ by the schema

$$\begin{cases} f(1) = c_1, \\ \quad \vdots \\ f(n) = c_n. \end{cases}$$

In particular, N_0 is at the same time the empty type $\emptyset$ and the logical constant falsehood $\perp$, and the function f of type $(\Pi x \in N_0)C(x)$ is the empty function. Similarly, N_1 is not only the one element type but also the logical constant truth $\top$.

N is a type, namely, the type of natural numbers. 1 is an object of type N and, if x is an object of type N, so is its successor x+1. Given an object c of type C(1) and a function g of type $(\Pi x \in N)(C(x) \to C(x+1))$ we may introduce a function f of type $(\Pi x \in N)C(x)$ by the recursion schema

$$\begin{cases} f(1) = c, \\ f(x+1) = g(x, f(x)). \end{cases}$$

Thinking of $C(x)$ as a proposition for every object x of type N, f is the proof of the universal proposition $(\Pi x \in N)C(x)$ which we get by applying the principle of mathematical <u>induction</u> to the proof c of $C(1)$ and the proof g of $(\Pi x \in N)(C(x) \rightarrow C(x+1))$.

To the axioms specified so far we may want to add axioms for certain types defined by transfinite induction. In the case of O, the type of ordinals of the second number class, these axioms run as follows. 1 is an object of type O. If x is an object of type O, so is $x+1$. If y is a sequence of objects of type O, that is, a function of type $N \rightarrow O$, then $y(1)+y(2)+\ldots$ is an object of type O. Finally, if c is an object of type $C(1)$, g a function of type $(\Pi x \in O)(C(x) \rightarrow C(x+1))$ and h a function of type $(\Pi y \in N \rightarrow O)((\Pi x \in N)C(y(x)) \rightarrow C(y(1)+y(2)+\ldots))$, then we may define a function f of type $(\Pi x \in O)C(x)$ by the schema of <u>transfinite recursion</u>

$$\begin{cases} f(1) = c, \\ f(x+1) = g(x, f(x)), \\ f(y(1)+y(2)+\ldots) = h(y, f \circ y), \end{cases}$$

where as usual

$$(f \circ y)(x) = f(y(x)).$$

By interpreting $C(x)$ as a proposition rather than a type, this schema gives us at the same time the principle of proof by <u>transfinite induction</u> over the second number class.

This finishes the informal description of the theory of types. In the formal theory the abstract entities (natural numbers, ordinals, functions, types, and so on) become represented by certain symbol configurations, called *terms*, and the definitional schemata, read from the left to the right, become mechanical *reduction rules* for these symbol configurations. A term *reduces* to another term if the latter can be obtained by repeatedly applying the reduction rules to parts of the former, and a term is *normal* if it cannot be further reduced, that is, if it has no part to which any of the reduction rules can be applied. For example, with the usual definitions of addition and multiplication,

$$\begin{cases} x+1 = x+1, \\ x+(y+1) = (x+y)+1, \end{cases} \qquad \begin{cases} x\cdot 1 = x, \\ x\cdot(y+1) = (x\cdot y)+x, \end{cases}$$

the term $(1+1)\cdot(1+1)$ reduces in three steps to the normal term $((1+1)+1)+1$. The normal terms have a very perspicuous form which can be determined by purely combinatorial reasoning. In particular, a normal numerical term is a *numeral*, that is, of the form

$$(\ldots((1+1)+1)+\ldots+1)+1,$$

and there is no normal term of type $\perp$.

The principal result of the proof theoretical analysis of the formal theory is the *normalization theorem* (proved in Martin-Löf 1971 for the fragment of the theory that is based solely on the axiom that there is a type of types and the axioms for cartesian products) which says that

every term reduces to a normal term.

Because of what was said at the end of the previous paragraph, the consistency of the formal theory follows combinatorially from the normalization theorem. Hence, according to Gödel's second incompleteness theorem, it cannot be proved without transcending the principles of reasoning that are formalized in the theory. What has turned out to be most expedient is to prove in the theory itself for an arbitrarily given term that it reduces to a normal term. The normalization theorem then follows by an application of the reflection principle

> if A(n) is provable in the theory for every numeral n, then A(x) for all natural numbers x.

The normalization theorem for the general intuitionistic theory of types provides us for the first time with an example of a number theoretic theorem (of the form $\forall x \exists y A(x,y)$ with A(x,y) primitive recursive) which we know how to prove intuitionistically, namely, by using the theory itself strengthened by the reflection principle, but which we do not know to be provable in classical set theory.

Formalization taken together with the ensuing proof theoretical analysis effectuates the computerization of abstract intuitionistic mathematics that above all Bishop 1967 has asked for. Suppose, for example, that we want to compute the value of a number theoretic function for a certain argument. We then find the term that denotes the function in the formal theory and apply it formally to the numeral that denotes the argument. The resulting term, which denotes the searched for value of the function, is a numerical term which, according to the normalization theorem, reduces to a numeral. This numeral is the final result of our computation. It only remains to remark that reducing a term to normal form is a mechanical process which can be implemented on a computer. Similarly, as soon as we have carried out the construction of a real number in the formal theory, we can program a machine to compute it with an arbitrary degree of approximation. What is doubtful at present is not whether it is possible to mechanize the abstract computations of intuitionistic mathematics, because we already know how to do that as a result of the proof theoretical analysis of formal intuitionistic theories, but rather whether these proof theoretical normalization procedures are at all useful for numerical computation. So far, they seem not to have found a single significant application.

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