CONSTRUCTIVE METRIC COMPLETION OF BOOLEAN ALGEBRAS

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Introduction

In this paper, we analyse some aspects of measure theory from a constructive point of view. We present first a simple and direct way to define the algebra of Lebesgue measurable subsets (quotiented by the ideal of sets of measure 0), that was actually advocated by Kolmogorov [8]. Classically, it is a remarkable fact that this Boolean algebra is complete, but constructively, it is only metrically complete, and not even σ-complete in general. We show then the constructive equivalence of this approach with the one of Bishop's school [2]. We analyse next the notion of Borel subsets. We show that the algebra of such subsets can be characterised in a pointfree and constructive way by an initiality condition. We then use our work to define in a purely inductive way the measure of Borel subsets, giving a solution to "Borel's measure problem" as formulated in Lusin's book [9]. As an illustration of the use of metrically complete Boolean algebras, we prove a constructive version of Kolmogorov's zero-one law.

1. Boolean rings with apartness relation

A commutative ring $A = (A; +, \cdot, 0)$ is *Boolean* if $x^2 = x$ for each $x \in A$. Observe that we do not assume that A has a unit 1, but if it has, we call A a *Boolean algebra*. We shall use the lattice-theoretic symbols $x \wedge y =_{\text{def}} xy$, $x \vee y =_{\text{def}} x + y + xy$ and $x \leq y$ for $x = x \wedge y$. The following useful inequalities hold in any boolean ring

$$(x + y) + (u + v) \leq (x + u) \vee (y + v). \tag{1}$$

$$xy + uv \leq (x + u) \vee (y + v). \tag{2}$$

Note that the subring

$$A_b = \{a \in A \mid a \leq b\}$$

of A is a Boolean algebra with b as unit, and $b \setminus x =_{\text{def}} x + xb$ as the complement of $x \in A_b$. If A is a Boolean algebra we write $\overline{x} =_{\text{def}} 1 \setminus x$.

A binary relation $\neq$ on A is a *strong apartness relation* if the following hold for all $x, y, z \in A$:

(A1) $\neg x \neq x$,
(A2) $x \neq y \implies y \neq x$,
(A3) $\neg x \neq y \implies x = y$,
(A4) $x \neq y \implies x + z \neq y + z$,

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(A5) $xy \neq 0 \implies x \neq 0$ and $y \neq 0$,
(A6) $x + y \neq 0 \implies x \neq 0$ or $y \neq 0$.

A *measure* on a Boolean ring A with strong apartness relation is a function $\mu : A \to [0, \infty)$ such that for all $x, y \in A$,

(M1) $\mu(x \vee y) = \mu(x) + \mu(y) - \mu(x \wedge y)$,
(M2) $\mu(x) > 0 \Longrightarrow x \neq 0$.

The measure is *positive* if for all $x \in A$,

(M3) $x \neq 0 \Longrightarrow \mu(x) > 0$.

A *measure ring* is a Boolean ring with strong apartness relation together with a positive measure. It is a *measure algebra* in case the ring is a Boolean algebra, and if in addition $\mu(1) = 1$, it is a *probability (measure) algebra.*

Proposition 1.1. *For any Boolean ring A and any function $\mu : A \to [0, \infty)$ satisfying $\mu(0) = 0$ and the identity (M1), a measure ring $\tilde{A} = (A/{\sim}, \neq, \mu)$ is defined from A by introducing the relations $x \neq y$ iff $\mu(x + y) > 0$ and $x \sim y$ iff $\neg x \neq y$. Moreover, $\tilde{A}$ is a measure algebra whenever A is a Boolean algebra.*

Proof. We have $x \sim y$ iff $\mu(x + y) = 0$. It follows from the monotonicity of μ, and the inequalities (1) and (2), that $\sim$ is a congruence with respect to addition and multiplication. Also μ respects $\sim$. (A1) follows from $\mu(0) = 0$. (A2-3) and (M1-3) are immediate from the definitions. (A4) is a consequence of $x + z + y + z = x + y$. (A5-6) follows easily using the monotonicity of μ. $\square$

Example 1.2. Write $[a, b)$ for the set of rational numbers q such that $a \leq q < b$. Let A be the set of subsets of $\mathbb{Q}$ of the form

$$x = [a_1, b_1) \cup \cdots \cup [a_n, b_n),$$

where $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n$, $n \geq 0$. The measure of x is $\mu(x) = \sum_{k=1}^{n} (b_k - a_k)$. Let $+$ be symmetric difference and $\cdot$ be intersection. This defines a Boolean ring A where μ satsifies $\mu(\emptyset) = 0$ and (M1). Then according to Proposition 1.1 we obtain a measure ring (which is not measure algebra).

If we restrict ourselves to intervals $[a, b)$ with $0 \leq a < b \leq 1$ we get a boolean algebra R. Notice that this Boolean algebra can be seen as the Boolean algebra generated by the atoms p_r for $0 \leq r \leq 1$ and the axioms

- $p_r \leq p_s$ if $r \leq s$
- $p_r = 1$ if $r \geq 1$
- $p_r = 0$ if $r \leq 0$

Example 1.3. Let C be the free Boolean algebra on the letters $x_1, x_2, \ldots$. For each i, let $\mu(x_i) = 1/2$. Extend μ to the whole of C by assuming that the letters represent independent events.

Lemma 1.4. *Let (A, μ) be a measure ring.*

(i) *$\rho(x, y) = \mu(x + y)$ defines a metric on A.*

(ii) *The operations and* $+$ *and* $\cdot$ *are uniformly continuous with respect to the metric* ρ.

Proof. (i): We have $\rho(x,y) = \rho(y,x)$, and $\rho(x,y) = 0$ if, and only if, $x + y = 0$ by axioms (A3) and (M2) for a measure algebra. The triangle inequality follows from (1).

(ii): This follows from the monotonicity of measure, and the inequalities (1) and (2). □

2. Completions of Boolean Rings

A sequence $(a_k)_{k=1}^{\infty}$ is *bounded* if there is some b such that $a_k \leq b$ for all k. A Boolean ring A is *boundedly σ-complete* if $\vee_{k=1}^{\infty} a_k$ exists for each bounded sequence $(a_k)_{k=1}^{\infty}$ in A. Note that in such a ring $\wedge_{k=1}^{\infty} a_k$ exists for each sequence $(a_k)_{k=1}^{\infty}$ in A. Clearly, in a Boolean algebra, every sequence is bounded, so in this case σ-complete is the same as boundedly σ-complete. Any measure ring (A,μ) can be equipped with a natural metric given by

$$\rho(x,y) = \mu(x+y).$$

Note that if a_n is a sequence which has a limit $\lim_n a_n$ in A with respect to ρ then

$$\mu(\lim_n a_n) = \lim_n \mu(a_n).$$

A measure ring is *metrically complete* if it is complete with respect to this natural metric.

Proposition 2.1. *Let (A,μ) be a metrically complete measure ring.*

(i) *If $\lim_n \vee_{k=1}^{n} x_k$ exists, then this limit is $\vee_{k=1}^{\infty} x_k$.*

(ii) *If $\lim_n \wedge_{k=1}^{n} x_k$ exists, then this limit is $\wedge_{k=1}^{\infty} x_k$.*

Lemma 2.2. *Let (A,μ) be a measure ring, and let (x_k) be a sequence in A bounded by b.*

(i) *If $\vee_{k=1}^{\infty} x_k$ exists, then $\wedge_{k=1}^{\infty}(b \setminus x_k)$ exists and $\wedge_{k=1}^{\infty}(b \setminus x_k) = b \setminus \vee_{k=1}^{\infty} x_k$.*

(ii) *If $\wedge_{k=1}^{\infty} x_k$ exists, then $\vee_{k=1}^{\infty}(b \setminus x_k)$ exists and $\vee_{k=1}^{\infty}(b \setminus x_k) = b \setminus \wedge_{k=1}^{\infty} x_k$.*

An (isometric) *embedding* f between measure rings (A,μ) and (B,ν) is a ring homomorphism which is measure preserving, i.e.

$$\nu(f(x)) = \mu(x) \qquad (x \in A).$$

Note that for such a map $x \neq y$ if, and only if, $f(x) \neq f(y)$. A ring homomorphism between measure rings (A,μ) and (B,ν) is *measurable* if for each $a \in A$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(\forall x \leq a)\,[\mu(x) \leq \delta \Rightarrow \nu(f(x)) \leq \varepsilon]. \tag{3}$$

Notice that since f is a ring homomorphism, $\nu(f(x+y)) = \nu(f(x) + f(y))$, this states that f is uniformly continuous on each subalgebra of the form A_a. The terminology will justified at the end of this section.

Theorem 2.3. *Let (A,μ) be a measure ring. Then there exists a metrically complete measure ring $(M_A, \overline{\mu})$ and an embedding $\phi : (A,\mu) \to (M_A, \overline{\mu})$, such that $\phi(A)$ is dense in M_A. If $\psi : (A,\mu) \to (B,\nu)$ is a measurable mapping into a metrically complete measure ring, then there exists a unique measurable mapping $\theta : (M_A, \overline{\mu}) \to (B,\nu)$ such that $\psi = \theta \circ \phi$. Furthermore, θ is an embedding, whenever ψ is an embedding.*

Proof. The proof follows closely that for a standard completion of a metric algebra. The ring M_A consists of Cauchy sequences (x_n) in (A, ρ). Its operations are defined termwise, and thus the algebraic identities are immediate. That M_A is closed under the operations $+$ and $\cdot$, follows from the lemma above. Two elements (x_n) and (y_n) of M_A are considered equal $(=)$ if for each natural number k there exists an m such that $\rho(x_n, y_n) \leq 2^{-k}$ for each $n \geq m$. They are apart $(\neq)$ if there exists k and m such that $\rho(x_n, y_n) > 2^{-k}$ for each $n \geq m$. It is now straightforward to check the apartness properties.

For any Cauchy sequence (x_k) in A, the sequence $(\psi(x_k))$ in B will be Cauchy. Since B is metrically complete we may define $\theta((x_k)) = \lim_k \psi(x_k)$. It is easily checked that this is a measurable homomorphism such that $\psi = \theta \circ \phi$. Uniqueness of θ follows by the fact that $\phi(A)$ is dense in M_A. Moreover, if ψ is an embedding $\nu(\theta((x_k))) = \nu(\lim_k \psi(x_k)) = \lim_k \nu(\psi(x_k)) = \lim_k \mu(x_k) = \overline{\mu}((x_k))$, so θ is an embedding as well.

□

The next result uses the non-constructive principle of limited omniscience (LPO). This principle is equivalent to the principle that each bounded monotone sequence of real numbers converges (cf. [10]). There exists however a constructive model of LPO together with dependent choice [5], which allows for an indirect interpretation of results based on these principles. Some other results will use the axioms of classical set theory (ZFC).

Theorem 2.4. *Let (A, μ) be a metrically complete measure ring.*

(i) *A is boundedly σ-complete, assuming LPO.*
(ii) *A is boundedly complete, assuming ZFC.*

Proof. (i): Suppose that (a_n) is a sequence of elements bounded by b. Then $b_n = a_1 \vee \cdots \vee a_n$ defines an increasing sequence (b_n) of elements bounded by b. Thus the sequence $(\mu(b_n))$ is convergent, according to LPO. Hence $\vee_{k=1}^{\infty} a_k$ exists.

(ii): Let $S \subseteq A_b$. Put $\alpha = \sup\{\mu(a) : a \in S\}$. Then choose a sequence (x_k) in S such that $\alpha - 1/k \leq \mu(x_k)$. By (i), $c = \vee_k x_k$ exists and $\mu(c) = \alpha$. Now, for any $a \in S$,

$$\mu(c) = \mu(a \vee c) = \mu((a \setminus c) \vee c) = \mu(a \setminus c) + \mu(c).$$

Hence $\mu(a \setminus c) = 0$, so $a \leq c$. Hence c is the supremum of S.

□

The next result uses a technique employed in connection with absolute continuity.

Theorem 2.5. *Let f be a ring homomorphism between the metrically complete measure rings (A, μ) and (B, ν). Then*

(i) *f measurable $\Longrightarrow$ f preserves the countable suprema that exist in A.*
(ii) *(ZFC) f preserves the countable suprema that exist in $A \Longrightarrow f$ measurable.*

Proof. (i) is straightforward. (ii): Suppose that f preserves the required suprema but that it is not locally uniformly continuous. Thus there is some $a \in A$, some $\varepsilon > 0$ and a sequence (x_k) in A, bounded by a, such that $\mu(x_k) < 2^{-k}$ and $\nu(f(x_k)) \geq \varepsilon$. Using countable subadditity we get

$$\mu(\bigvee_{k=n}^{\infty} x_k) \leq \sum_{k=n}^{\infty} \mu(x_k) < 2^{-n+1}$$

and

$$\nu(f(\bigvee_{k=n}^{\infty} x_k)) \geq \nu(f(x_k)) \geq \varepsilon.$$

Since the sequence (x_k) is bounded, $x = \wedge_{n\geq 1} \vee_{k\geq n} x_k$ exists. Then using the fact that f preserves countable suprema, and hence also countable infima, we get

$$\nu(f(x)) = \nu(\bigwedge_{n=1}^{\infty} f(\bigvee_{k=n}^{\infty} x_k)) = \lim_n \nu(f(\bigvee_{k=n}^{\infty} x_k)) \geq \varepsilon.$$

Moreover $\mu(x) = 0$, so $x = 0$. Hence $\nu(f(x)) = \nu(f(0)) = \nu(0) = 0$, which is a contradiction.

□

Let A be the Boolean algebra of decidable subsets of $\{1, 2, \ldots, n\}$ and let μ be the measure given by $\mu(\{i\}) = 1/n$. Then any ring homomorphism $f : (A, \mu) \to (B, \nu)$ becomes measurable by taking $\delta = 1/2n$.

3. Kolmogorov's zero-one law

Two elements (*events*) a and b of a probability algebra (A, μ) are called *independent* if $\mu(a \wedge b) = \mu(a)\mu(b)$. Two sets of events S and T are *independent* if a and b are independent for all $a \in S$ and $b \in T$. A finite sequence of events $a_1, \ldots, a_n$ is *independent* if for every $f : \{1, \ldots, n\} \to \{0, 1\}$

$$\mu(\bigwedge_{i=1}^{n} a_i^{f(i)}) = \prod_{i=1}^{n} \mu(a_i^{f(i)}).$$

An infinite sequence of events (a_n) is *independent* if every finite initial segment is independent. These notions extend to sequences of sets of events in the obvious way.

A *multiplicative set* M in a Boolean algebra is a set, containing 1, which is closed under multiplication. For any subset T of a Boolean algebra, let $M(T)$ be the set of finite products $a_1 \cdots a_n$, where $n \geq 0$ and $a_i \in T$. This is the smallest multiplicative set containing T. For any multiplicative set M in a Boolean algebra A define $S(M)$ to be the set of events in A of the form $a_1 + \cdots + a_n$, where $a_1, \ldots, a_n \in M$ and $n \geq 0$. Clearly, this is the smallest Boolean subalgebra of A containing M.

Lemma 3.1. *Let $M_1, \ldots, M_n$ be independent multiplicative subsets of a probability algebra A. Then $S(M_1), \ldots, S(M_n)$ are independent as well.*

Proof. Let $S_k(M)$ be the set of events that can be written as a sum of at most k events from M. We first prove that for $i \neq j$, M_i is independent from $S_k(M_j)$ by induction on k.

For $k = 0, 1$, this is trivial. Suppose that $a \in M_i$, $b \in S_k(M_j)$ and $c \in M_j$. Then using finite additivity and the induction hypothesis,

$$\begin{aligned} \mu(a(b+c)) &= \mu(ab) + \mu(ac) - 2\mu((ac)b) \\ &= \mu(a)\mu(b) + \mu(a)\mu(c) - 2\mu(ac)\mu(b) \\ &= \mu(a)\mu(b) + \mu(a)\mu(c) - 2\mu(a)\mu(c)\mu(b) \\ &= \mu(a)\mu(b+c). \end{aligned}$$

Next, we prove that the sets $S_{k_1}(M_1), \dots, S_{k_n}(M_n)$ are independent, for all $(k_1, \dots, k_n) \in \mathbb{N}^n$. We do this by induction on $\mathbb{N}^n$ with the product order $\leq_n$, which is a wellfounded relation. We have $S_0(M) = \{0\}$, so the proof is trivial for any vector $\mathbf{k} = (k_1, \dots, k_n)$ containing a zero. Thus assume $\mathbf{k} \geq_n (1, \dots, 1)$. Consider $a_i \in S_{k_i}(M_i)$, $i = 1, \dots, n$. By finite additivity and the inductive assumption $\mu(a_1 \cdots a_{i-1}(a_i + b)a_{i+1} \cdots a_n)$ equals

$$\begin{aligned} & \mu(a_1 \cdots a_n) + \mu(a_1 \cdots a_{i-1} b a_{i+1} \cdots a_n) - 2\mu(a_1 \cdots a_{i-1}(a_i b) a_{i+1} \cdots a_n) \\ = \ & \mu(a_1) \cdots \mu(a_n) + \mu(a_1) \cdots \mu(a_{i-1})\mu(b)\mu(a_{i+1}) \cdots \mu(a_n) \\ & -2\mu(a_1) \cdots \mu(a_{i-1})\mu(a_i b)\mu(a_{i+1}) \cdots \mu(a_n). \end{aligned}$$

By the first result we have $\mu(a_i b) = \mu(a_i)\mu(b)$ and thus the righthand side equals

$$\mu(a_1) \cdots \mu(a_{i-1})\mu(a_i + b)\mu(a_{i+1}) \cdots \mu(a_n).$$

□

Lemma 3.2. *Let $S_1, \dots, S_n$ be independent subsets of a metrically complete probability algebra A. Then their metric closures $\underline{S_1}, \dots, \underline{S_n}$ are again independent.*

Proof. This follows readily from the continuity of $\wedge$ with respect to the natural metric. □

For a subset T of a metrically complete measure algebra A, define $\sigma(T)$ to be $\underline{S(M(T))}$, i.e. the metric closure of the least Boolean algebra containing T. The following result is now a direct consequence of Lemma 3.1 and 3.2.

Proposition 3.3. *Let A be a metrically complete probability algebra, and let $M_1, M_2, M_3, \dots$ be a sequence of independent multiplicative subsets of A. Then $\sigma(M_1), \sigma(M_2), \sigma(M_3), \dots$ are independent as well.*

For any sequence (a_n) of events in a metrically complete probability algebra, define the set of *tail events*

$$\mathcal{T}((a_n)) = \bigcap_{n=1}^{\infty} \sigma(\{a_n, a_{n+1}, a_{n+2}, \dots\}).$$

The proof of Kolmogorov's zero-one law below follows Billingsley [4].

Theorem 3.4. *Let (A, μ) be a metrically complete probability algebra and let (a_n) be a sequence of independent events. Then for every $b \in \mathcal{T}((a_n))$ either $\mu(b) = 0$ or $\mu(b) = 1$.*

Proof. Let b be an arbitrary element in $\mathcal{T} = \mathcal{T}((a_n))$. One shows easily that $M_1 = M(\{a_1\}), \ldots, M_n = M(\{a_n\})$ and $M_{>n} = M(\{a_{n+1}, a_{n+2}, \ldots\})$ are independent sets. Hence, by Proposition 3.3, $\sigma(M_1), \ldots, \sigma(M_n), \sigma(M_{>n})$ are independent. Since $\mathcal{T} \subseteq \sigma(M_{>n})$, this shows that $b, a_1, \ldots, a_n$ are independent. Since n was arbitrary, the infinite sequence $b, a_1, a_2, \ldots$ is in fact independent. As above one then sees that $M(\{b\})$ and $M_{>n}$ are independent, regardless of n. Thus $M(\{b\})$ and $\mathcal{T}$ are independent. Since b is in both of these sets, it is independent of itself! Thus $\mu(b) = \mu(b^2) = \mu(b)^2$, and hence $\mu(b) = 0$ or $\mu(b) = 1$. □

Billingsley [4, p. 63] writes concerning the classical theorem: "Kolmogorov's result is surprisingly general, and it is in many cases quite easy to use it to show that the probability of some set must have one of the extreme values 0 and 1. It is perhaps curious that it should so often be very difficult to determine which of these extreme values is the right one." By our constructive theorem one can indeed pick the correct value. However, it might be more difficult to prove (constructively) that a particular event is a tail event.

4. Integrable sets

Bishop and Cheng [2] (see also [1]) developed constructive measure theory based on a version of the Daniell integral. We show that the integrable sets of this approach form a metrically complete measure ring.

Let X be a set with an apartness relation $\neq$. A pair $A = (A_1, A_2)$ of subsets of X is called a *complemented set in* X, if $x \neq y$ whenever $x \in A_1$ and $y \in A_2$. Operations on complemented sets are defined as follows ([2]):

$$\begin{aligned}
(A_1, A_2) \wedge (B_1, B_2) &= (A_1 \cap B_1, (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (A_2 \cap B_2)), \\
(A_1, A_2) \vee (B_1, B_2) &= ((A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1), A_2 \cap B_2), \\
\overline{(A_1, A_2)} &= (A_2, A_1).
\end{aligned}$$

Then define $A \setminus B = A \wedge \overline{B}$ and $A + B = (A \setminus B) \vee (B \setminus A)$. A computation shows

$$(A_1, A_2) + (B_1, B_2) = (A_1 \cap B_2 \cup A_2 \cap B_1, A_1 \cap B_1 \cup A_2 \cap B_2).$$

Let $\mathbf{0}_X = (\emptyset, X)$ and $\mathbf{1}_X = (X, \emptyset)$. Let $\mathcal{K}(X) = (\mathcal{K}(X), +, \wedge, \mathbf{0}_X, \mathbf{1}_X)$

A *semiring* is a ring whose additive structure is a commutative monoid. A *Boolean semiring* is a commutative semiring satisfying $a^2 = a$ for all a. The following result is verified by straightforward calculations.

Lemma 4.1. *The set $\mathcal{K}(X)$ of complemented sets in X forms a Boolean semiring with unit.*

Note, however, that $\mathcal{K}(X)$ is not even a lattice, since the absorption identity $A \wedge (A \vee B) = A$ fails: take $A = (X, \emptyset)$ and $B = (\emptyset, \emptyset)$.

Consider an integration space (X, L, I) and its completion (X, L_1, I_1) (see [1]). Recall that a complemented set $A = (A_1, A_2)$ in X is *integrable* if the (partial) characteristic function χ_A, which is 1 on A_1 and 0 on A_2, is an integrable function. The *measure* of A is

$\mu(A) = I_1(\chi_A)$. It can be shown [1] that if A and B are integrable sets in X, then so are $A \wedge B$, $A \vee B$, $A \setminus B$ and hence $A + B$. Thus we may define the relations

$$\begin{aligned} A \sim B &\Leftrightarrow_{\mathrm{def}} \mu(A+B) = 0, \\ A \neq B &\Leftrightarrow_{\mathrm{def}} \mu(A+B) > 0. \end{aligned}$$

Obviously, the negation of $A \neq B$ is equivalent to $A \sim B$. Let $\mathcal{I}(X)$ be the set of integrable sets in X equipped with $\sim$ as equality and $\neq$ as apartness.

Theorem 4.2. *For each complete integration space (X, L_1, I_1), the algebra of integrable sets $\mathcal{I}(X) = (\mathcal{I}(X); +, \cdot, \mathbf{0}_X; \mu)$ is a metrically complete measure ring.*

Proof. To verify the axioms for a Boolean ring we need, according to Lemma 4.1, only to check that $\sim$ is a congruence and that additive inverses exist. Then according to Proposition 1.1 we need to check that $\sim$ respects μ, that $\mu(\mathbf{0}_X) = 0$ and that the identity (M1) holds. This then proves that $\mathcal{I}(X)$ is measure algebra. That it is metrically complete follows by an observation (Problem 6.2) in [1].

We fill in the above outline:

According to Lemma 6.2.2 – 6.2.3 of [1], an integrable set $A = (A_1, A_2)$ has measure 0 iff A_2 is a full set. Hence, $A \sim B$ iff $A_1 \cap B_1 \cup A_2 \cap B_2$ is full. Using this characterisation it is easy to check the congruence properties of $\sim$.

Now $A + A = (\emptyset, A_2 \cup A_1) \sim (\emptyset, X) = \mathbf{0}_X$, since $A_2 \cup A_1$ is full for any integrable A ([1]).

The identity (M1) is Proposition 6.3.5 in [1]. □

We now show that the point-free versions of measure algebras arising from metric completions of the algebras R and C, of Examples 1.2 and 1.3, coincide with those obtained as integration spaces.

Let $\phi : R \to M_R$ be the metric completion according to Theorem 2.3. Let $X = [0, 1]$ be the unit interval of $\mathbb{R}$, let $L = C([0, 1])$ be the set of all continuous functions $[0, 1] \to \mathbb{R}$ and let $I(f) = \int_0^1 f(x)dx$ be the Riemann integral. This defines an integration space (X, L, I), whose completion (X, L_1, I_1) is the Lebesgue integral on $[0, 1]$ (see [1]). Now define an embedding $\psi : R \to \mathcal{I}([0, 1])$ by letting $\phi([a_1, b_1) \cup \cdots \cup [a_n, b_n))$ be the integrable set

$$((a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n), [0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_n, 1]).$$

By Theorem 2.3 it follows that there is a unique embedding $\theta : M_R \to \mathcal{I}([0, 1])$ such that $\psi = \theta \circ \phi$. As in Exercise 6.17 of [1] one shows that $\phi(R)$ is dense in $\mathcal{I}([0, 1])$ thus proving that θ is an isomorphism.

A similar argument can be made in the case of the Cantor space C, taking $X = 2^{\mathbb{N}}$ and L to be the functions $X \to \mathbb{R}$ that are represented by finite binary trees, with real numbers as leaves.

5. A Point-Free Approach to Borel Sets

The problem we analyse in this section is the following: given a point-free description of a topological space X we ask for a point-free description of the σ-algebra $Borel_X$ of the Borel subsets of X. We solve this problem in the case where X is the Cantor space or $X = [0, 1)$. We show classically that this definition is equivalent to the usual one, but in each case, the point-free description uses only inductive definitions and intuitionistic logic.

5.1. σ-Completion of a Boolean algebra.

Theorem 5.1. *For any Boolean algebra A there exists a σ-complete Boolean algebra B_A and an embedding $i : A \to B_A$, such that $i(A)$ generates B_A. Moreover, if $f : A \to B$ is any other map from A to a σ-complete Boolean algebra B there exists a unique σ-algebra morphism $\bar{f} : B_A \to B$ such that $\bar{f} \circ i = f$.*

Proof. What is not direct in this statement is that the map $i : A \to B_A$ is an embedding, i.e. satisfies $i(x) \leq i(y)$ iff $x \leq y$. For this, we can adapt the explicit description of B_A presented in [11] in the special case where $A = C$ (cf. Example 1.3). We consider infinitary propositional formulae over A

$$\phi \;=\; \bigvee_n \phi_n \mid \bigwedge_n \phi_n \mid a \in A$$

We define negation $\neg\phi$ of formulae recursively using the de Morgan laws. We let *sequents* be finite set of formulae and we have the following derivation rules for sequents

$$\frac{a_1 \vee \cdots \vee a_n = 1}{\vdash a_1, \ldots , a_n, \Gamma}$$

$$\frac{\bigvee_n \phi_n \in \Gamma \qquad \vdash \phi_{n_0}, \Gamma}{\vdash \Gamma}$$

$$\frac{\cdots \vdash \phi_n, \Gamma \cdots}{\vdash \bigwedge_n \phi_n, \Gamma}$$

The main lemma is then that $\vdash \Gamma$ follows from $\vdash \Gamma, \phi$ and $\vdash \Gamma, \neg\phi$ (cut-elimination, [11]). We define recursively $[\![\phi]\!] \in B_A$, by taking $[\![a]\!] = i(a)$, and, using cut-elimination, we can show that

$$\vdash \phi_1, \ldots, \phi_n$$

iff $\vee [\![\phi_i]\!] = 1 \in B_A$. ∎

Notice that the inclusion $i : A \to A_1$ however does not preserve, in general, countable sup that may exist in A.

The importance of this construction is that it allows us to define in a point-free way the σ-algebra of Borel subsets of a totally disconnected, compact Hausdorff space X: in a point-free way this space is defined as the Boolean algebra A of its closed-open subset and we take B_A to be the σ-algebra of Borel subsets of X. We show later how to give a similar point-free definition of Borel subsets of the space $[0, 1)$, but first we shall justify this definition classically.

5.2. **Equivalence with the usual definition.** If X is a topological space, we let $Borel_X$ be the σ-algebra of Borel subsets of X. If A is a Boolean algebra, and X be the associated space of ultrafilters of B [7], any element $a \in A$ defines a closed open subset $S(a) \subseteq X$. We recall that a *meager* set in a topological space is a subset of a countable union of closed subset of empty interior, and Baire's theorem says that a closed open subset $S(y)$ is meager iff $y = 0 \in A$.

Let B be any σ-complete Boolean algebra and Y be the associated space of ultrafilters of B.

Lemma 5.2. *(ZFC) There is one and only one σ-algebra map $h : Borel_Y \to B$ such that the symmetric difference $T + S(h(T))$ is a meager set of Y for any $T \in Borel_Y$.*

Proof. We consider the set M of $T \in Borel_Y$ such that there exists $y \in B$ such that $T + S(y)$ is meager. Notice that such a y is uniquely determined because, by Baire's category theorem, a meager closed open set in Y is empty: if both $S(y)+T$ and $S(y_1)+T$ are meager then so is $S(y) + S(y_1) = S(y + y_1)$ and hence $y + y_1 = 0$ that is $y = y_1$.

Clearly M contains each set $S(y)$ for $y \in B$. Furthermore M is closed under unions because if $y_n \in B$ then the difference $S(\bigvee y_n) - \bigcup S(y_n) \subseteq Y$ is a closed subset of empty interior. Hence $M = Borel_Y$ and for each $T \in Borel_Y$ there exists $h(T) \in B$ such that $T + S(h(T))$ is meager. By construction h is a σ-algebra map. □

Notice that we have $h(S(y)) = y$ if $y \in B$ since y is the unique element y_1 of B such that $S(y) + S(y_1)$ is meager.

Theorem 5.3. *(ZFC) Let A the Boolean algebra of closed open subsets of a separable totally disconnected compact Hausdorff space X. Then the σ-algebra $Borel_X$ is a σ-completion B_A of A.*

Proof. Let B be a σ-algebra and f a morphism of Boolean algebras $f : A \to B$. Let Y be the space of ultrafilters of B. The map f corresponds canonically to a continuous map $g : Y \to X$ [7], and the map $T \longmapsto g^{-1}(T)$ defines a map $f_1 : Borel_X \to Borel_Y$. By the lemma we have a retraction $h : Borel_Y \to B$ and the composition $h \circ f_1$ is then the (unique) extension of f to $Borel_X$. □

This theorem is a slight variation of a result of [7], Theorem 14 p. 102, stated there in the case where X is of the form 2^I for some set I. In [7] this result is seen as a possible construction of a free σ-complete Boolean algebra; we see it here as a justification of a purely inductive approach to the description of Borel sets.

5.3. **Completeness Theorem and Borel subsets of Cantor Space.** The first application of this result is the completeness theorem of propositional ω-logic. Let B be a free σ-algebra on letters $x_1, x_2, \ldots$ This is a σ-completion of the algebra C of Example 1.3. Any element $a \in C$ defines a closed open subset $S(a)$ of Cantor space $X = 2^N$ by taking $S(x_i)$ to be the set of $\alpha \in X$ such that $\alpha(i) = 1$. Let $\phi, \psi, \ldots$ be infinitary propositional formulae described by the syntax

$$\phi \;=\; \bigvee_n \phi_n \mid \bigwedge_n \phi_n \mid x_i \mid x_i'$$

Each such formula ϕ has a canonical interpretation as an element $\phi \in B$ and as a Borel subset $[\![\phi]\!] \in Borel_X$, where $[\![\phi]\!]$ is the set of truth assignments $\alpha \in X$ that makes the formula ϕ true:

- $[\![x_i]\!] = S(x_i)$
- $[\![x'_i]\!] = X - S(x_i)$
- $[\![\bigvee_n \phi_n]\!] = \cup[\![\phi_n]\!]$
- $[\![\bigwedge_n \phi_n]\!] = \cap[\![\phi_n]\!]$

Theorem 5.4. *(ZFC) We have $\phi \leq \psi \in B$ iff $[\![\phi]\!] \subseteq [\![\psi]\!]$ in $Borel_X$.*

Proof. This is a reformulation of the previous theorem. □

This shows that we can define in a point-free way the algebra of Borel subsets of Cantor space as the free σ-algebra on letters $x_1, x_2, \ldots$

5.4. Borel subsets of $[0, 1)$.

Theorem 5.5. *The σ-algebra of Borel subsets of $[0, 1)$ is the σ-algebra generated by p_r and the relations*

- $p_r = \bigvee_{s<r} p_s$
- $p_r = 0$ *if* $r \leq 0$
- $p_r = 1$ *if* $r \geq 1$

Proof. Intuitively, p_r represents the proposition $0 \leq x < r$. A model of these axioms defines exactly a real number in $[0, 1)$.

We consider infinitary formulae described by the syntax

$$\phi \;=\; \bigvee_n \phi_n \mid \bigwedge_n \phi_n \mid p_r \mid p'_r$$

We can define in the usual way binary conjunction, disjunction and negation of such formulae from these basic connectives. Any such formula ϕ represents a Borel subset $[\![\phi]\!] \subseteq [0, 1)$ interpreting p_r as $[0, r)$.

Let T be the formula representing the conjunction of the relations in the statement of the theorem. We have $[\![\phi]\!] \subseteq [\![\psi]\!]$ iff any model of $\phi \wedge T$ is a model of ψ iff, by Theorem 5.4, $\phi \wedge T \leq \psi$ in the free σ-algebra on p_r iff $\phi \leq \psi$ in the σ-algebra generated by p_r and the relations above. □

Another equivalent inductive description of this σ-complete Boolean algebra is the σ-completion of the algebra R of Example 1.2 quotiented by the relation $p_r = \bigvee_{s<r} p_s$.

6. Measure of Borel Sets

We use the previous characterisation of Borel sets to define a measure on Borel sets, first assuming LPO and then in a constructive way, but defining the measure as an hyperarithmetical real.

6.1. **Measure of Borel Sets, assuming LPO.** We recall that Borel original definition of measure was purely inductive [3], by induction on the construction of the Borel set (what Borel called first "measurable" and then "well-defined" set). This suggests a "coherence" problem: to show that this definition depends only on the set itself and not on its construction as a Borel set. To give an inductive solution of this problem was called by Lusin [9] "Borel's measure problem". We present here such a solution, assuming LPO. For a constructive justification of such use of LPO, see [5].

Let B_C be the σ-completion of C and M_C be the metric completion of C, μ (cf. Example 1.3). With *LPO*, the algebra M_C is σ-complete, by Theorem 2.4, and hence there is a unique map $B_C \to M_C$ preserving the letters $x_1, x_2, \ldots$ By composing this map with the measure map $\mu : M_C \to [0,1]$ we get a measure on B_C.

The situation is a little more complex for $[0,1)$. Let B_R be the σ-algebra built freely on R (cf. Example 1.2) with the relation $p_r = \bigvee_{s<r} p_s$, and M_R be the metric completion of R, μ. In order to define a map $B_R \to M_R$ the only thing to check is given by the following

Proposition 6.1. *In the metric completion of R, μ we have $p_r = \bigvee_{s<r} p_s$*

Proof. We have $p_s \leq p_r$ if $s \leq r$. Also, let r_n be a sequence of rationals $< r$ converging to r. Then (p_{r_n}) is a Cauchy sequence converging to p_r. Hence $p_r = \bigvee p_{r_n}$ by Proposition 2.1. □

6.2. **Measure of Borel Sets as Hyperarithmetical Reals.** It is well-known that constructively, even open sets of Cantor space may fail to have a (computable) measure (see [11]). However, constructively, there is the option of defining the value of the proposition $r < \mu(x)$, for r rational and $x \in B_C$ as an *hyperarithmetical proposition.* Such a proposition is defined constructively as an element of the free σ-complete Boolean algebra H. The following theorem can be stated *and* proved intuitionistically, without assuming LPO.

Theorem 6.2. *It is possible, in a unique way, to assign a value $f(x,r) \in H$, written $f(x,r) = [\mu(x) < r]$, for $x \in B_C,\ r \in Q$, so that*

- $[\mu(x_n) < r] = [1/2 < r]$
- $[\mu(x'_n) < r] = [1/2 < r]$
- $[\mu(\bigwedge_n u_n) < r] = \bigvee_n [\mu(\bigwedge_{m \leq n} u_m) < r]$
- $[\mu(\bigvee_n u_n) < s] = \bigvee_{r<s} \bigwedge_n [\mu(\bigvee_{m \leq n} u_m) < s]$

Proof. We use a variation of the Cauchy completion of C. First we define inductively a class of directed posets: we include

- the singleton $\{0\}$,
- the product $I \times J$ of any already built I and J,
- if we have already built a sequence I_n of directed posets, we can form *sup* (I_n) which is the set of all finite sequences $(i_0, \ldots, i_{n-1})$, $i_k \in I_k$ with the order

$$(i_1, \ldots, i_{n-1}) \leq (j_1, \ldots, j_{m-1})$$

defined by $n \leq m$ and $i_k \leq j_k$.

We consider then the set G of generalised sequences $(x_i)_{i\in I}$ indexed over such an indexed set, with values $x_i \in C$. The operations $\vee, \cdot$ are defined by

$$(x_i) \vee (y_j) = (x_i \vee y_j) \qquad (x_i) \cdot (y_j) = (x_i \cdot y_j)$$

while we have

$$\bigvee (x_{ni})_{i\in I_n} = (x_{0i_0} \vee \cdots \vee x_{(n-1)i_{n-1}})_{(i_0,\ldots,i_{n-1})\in sup\ (I_n)}$$

We can define $\mu((x_i)) < r$ to be $\bigvee_{i_0} \bigwedge_{i_0\leq i} [\mu(x_i) < r]$.

We consider next the subset M of Cauchy sequences, i.e. sequences (x_i) satisying

$$1 = \bigwedge_{r>0} \bigvee_{i_0} \bigwedge_{i\geq i_0, j\geq i_0} \mu(x_i + x_j) < r$$

This set can be seen as a constructive form of the Cauchy completion of C and it can be checked that M is a σ-complete boolean algebra. We then have a unique mapping of σ-algebras $B_C \to M$ which can be used to define $\mu(x) < r$ for $x \in B_C$. □

Related Work

A direct approach of measurable functions and subsets via Cauchy completion has been suggested by several people, for instance by A. Weil [14]. A natural question to be explored is if one should start from basic subsets, as in the present work, or from a given dense subalgebra of continuous functions, as suggested by I. Segal [12, 13].

References

[1] E. Bishop and D.S. Bridges. *Constructive Analysis.* Springer 1985.

[2] E. Bishop and H. Cheng. *Constructive Measure Theory.* Mem. Amer. Math. Soc. 116 (1972).

[3] E. Borel. *Leçons sur la Théorie des Fonctions.* Gauthier-Villars, quatrième édition 1950, 1ere édition 1898.

[4] P. Billingsley. *Probability and Measure.* Third ed. Wiley 1995.

[5] Th. Coquand and E. Palmgren. Intuitionistic choice and classical logic. *Archive for Mathematical Logic* 39(2000), pp. 53 – 74.

[6] P.R. Halmos. *Measure Theory.* Springer 1974.

[7] P.R. Halmos. *Lectures on Boolean algebras.* Princeton, 1963.

[8] A.N. Kolmogorov. Complete metric Boolean algebras. *Philosophical studies* 77(1995), pp. 57 – 66. (Translation of "Algèbres de Boole métrique complètes" VI Zjazd Mathematyków Polskich, Warsaw, 1948, pp. 21 – 30.)

[9] N. Lusin. *Leçons sur les ensembles analytiques.* Gauthier-Villars, 1930.

[10] M. Mandelkern. Limited omniscience and Bolzano-Weierstrass principle. *Bull. London Math. Soc.* 20(1988), pp. 319 – 320.

[11] P. Martin-Löf. *Notes on Constructive Mathematics.* Almqvist and Wiksell, Stockholm, 1968.

[12] I. Segal. Algebraic integration theory. Bull. Amer. Math. Soc. 71 (1965), pp. 419–489.

[13] I. Segal. Abstract probability spaces and a theorem of Kolmogoroff. Amer. J. Math. 76 (1954), pp. 721 – 732.

[14] A. Weil. Calcul des probabilités, méthode axiomatique, intégration. *Revue Sci.* (Rev. Rose Illus.) 78(1940), pp. 201 – 208.

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