Some results about Measure Theory

Chalmers University

Preliminary version, August 1999

We present some fundamental results/definitions on measure theory, that should be in any book on this topic.

1 Borel Sets

The *original* definition of Borel sets and Borel measure is much clearer than any other later presentation. To simplify Borel consider only subsets of the open interval $(0,1)$. A set is *well-defined* (now called Borel sets) iff it can be generated by using the rules

- (r,s) is well-defined,
- if we have a sequence of well-defined and *disjoint* sets A_n then $\cup A_n$ is well-defined,
- if $A \subseteq B$ are well-defined then so is $B - A$.

Then on these sets, Borel defines what should be the measure $\mu(A) \in [0,1]$:

- $\mu(r,s) = s - r$,
- $\mu(\cup A_n) = \Sigma \mu(A_n)$,
- $\mu(B - A) = \mu(B) - \mu(A)$.

We have then a clear problem: to show that this definition is *consistent*. That is, if $A = A'$ then $\mu(A) = \mu(A')$. This was solved by Lebesgue, but his solution involves the consideration of *arbitrary* subset of the reals. One can wonder if a direct solution, considering only well-defined sets, can be given. This is known as (cf. Lusin's book) as *Borel measure problem.* (Borel sketched a solution in one later edition of his book on real functions.)

Notice that this definition is equivalent to the usual one: one can show by induction first that the intersection of two well-defined sets are well-defined, and then that the union of *any* sequence of well-defined sets is well-defined.

But the usual definition looks rather arbitrary, while Borel definition is motivated by the requirement on the measure of a well-defined set.

2 Ulam's matrix

Let Ω be the first uncountable ordinal. Then there is *no* non trivial measure on the set of all subsets of Ω. For this let $i_a : [0,a[\to N$ be a one-to-one map from $[0,a[$ to N for $a \in \Omega$. Define

$$S_{n,b} = \{a \in \Omega \mid b < a \wedge i_a(b) = n\}.$$

If $b_1 < b_2$ then S_{n,b_1} and S_{n,b_2} are disjoint: indeed because i_a is one-to-one, we cannot have $i_a(b_1) = i_a(b_2) = n$.

Furthermore $\cup_n S_{n,b} = \{a \mid b < a\}$ has a complement which is countable.

If we have a measure μ such that $\mu(\Omega) = 1$ and $\mu(\{a\}) = 0$ for all $a \in \Omega$ then we have $\mu(\cup_n S_{n,b}) = 1$. Hence for all b there exists n such that $\mu(S_{n,b}) > 0$.

But then, since Ω is *not* countable there exists a fixed n_0 such that $\mu(S_{n_0,b}) > 0$ for uncountably many b. This is impossible since all $S_{n_0,b}$ are disjoint.

A corollary: with the continuum hypothesis, it is impossible to have a measure for all subsets of $[0,1]$.

3 Ultrafilter

This is a result of Sierpinski, that if we have a non principal ultrafilter then we have a non measurable subset. We consider the boolean algebra $X = F_2^N$. We assume to have a non trivial boolean map $\mu : F_2^N \to F_2$. I claim then that $A = \{x \in F_2^N \mid \mu(x) = 1\}$ is not measurable.

Indeed, if it is measurable, by symmetry, it has measure $1/2$ and its measure is > 0. But it is a basic result on Haar measure that this implies that $A - A = \{x - y \mid x, y \in A\}$ contains then a nonempty open subset: indeed the function $x \longmapsto \chi_{A+x}\chi_A$ is continuous from X to $L^1(X)$ and hence so is

$$\phi : x \longmapsto \int \chi_{A+x}\chi_A dm = m(A \cap (A + x)).$$

Since $\phi(0) > 0$ we have $\phi(x) > 0$ on a neighborhood of 0. In particular there exists N such that $u \in A - A$ if $u(i) = 0,\ i < N$. But then, if we fix $x_0 \in A$ all $x_0 + u$ belongs to A and A contains all sequences x such that $x(i) = x_0(i),\ i < N$. Also, we have $\mu(x) = \mu(y)$ if $x(n)$ and $y(n)$ differs only on finitely many n. This implies $A = X$, which contradicts $\mu(0) = 0$.

4 Haar measure on compact groups

The construction of a mean by von Neumann is quite elegant. Let G a compact group, and f a continuous function on G. We want to define its mean value. We take the compact convex closure of the set of all finite average of left translate of f. The function $g \longmapsto max(g)$ is continuous on this set, and hence has a minimum $I(f)$. This minimum is the mean of f. It is clear by construction that the mean of f is the same as the mean of any left translate of f.

To show that this is what we expect: we take any g such that $max(g) = I(f)$ and we show that g is constant: this is clear for any average of g has the same maximum value. Hence we can approximate $I(f)$ by suitable average of left translated of f, and we can define $I(f)$ as *the* constant which can be approximated by average of left translate of f. We then prove that we get the same value if we take right translate.

It follows from this that $I(f + g) = I(f) + I(g)$.