A new model of Dependent Type Theory

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Thanks to Marc for so many fruitful discussions!

Some works with Marc

We met in 1993; knew his name from his work on constructive Ramsey theorem and his work on *strongly majorizable functionals*

Some works with Marc

A realization of the negative interpretation of the Axiom of Choice, 1993

Automating coherent logic, 2003

A Kripke model for simplicial sets, 2013

A model of type theory in cubical sets, 2013

Syntactic Forcing Models for Coherent Logic, 2017

Loop-checking and the uniform word problem for join-semilattices with an inflationary endomorphism, 2022

Type Theory with explicit universe polymorphism, 2023

Cubical set models

Exactly 10 years old

Found while trying to simplify Richard Williamson's concrete presentation of cubical sets in his lecture notes on *Combinatorial homotopy theory*

Cubical set models

One basic intuition: homogeneity of space, no direction should be privileged

Hence *symmetric* cubical set

Sjoerd Crans had a similar presentation of cubical sets, but with a linear ordering, which breaks the symmetry

When I showed this to Andy Pitts he pointed out the connection with *nominal sets* and more precisely, nominal sets with *name-restrictions* (for modelling local scoping of names), that were discovered by Sam Staton to form a presheaf topos

Exercise 9.7 in the book of Andy Pitts on nominal sets

Cubical set model

From this (unexpected) connection with nominal λ-calculus, it follows that this interpretation has a clear computational aspect

Dependent type theory based on λ-calculus

For making computational sense of univalence, it seems that we need such a nominal extension

All the constructivity problems with the simplicial set model disappear magically!

Problem with Higher Inductive Types

While trying to implement elimination rule for the circle, seen as a Higher Inductive Type, we found out (January 2014) that we naturally need *diagonals*

This also corresponds to a variation of nominal sets

Nominal renaming sets, M. Gabbay, M. Hofmann, 2008

This lead to a variation of the model, which applies to a large class of presheaf models

(Not to the simplicial set model however, the interval has to be *tiny*, but it provides a simplified presentation of this model in a classical metatheory)

Problem with Higher Inductive Types

Also this lead to a semantics of parametrised Higher Inductive Types such as suspension

The main idea in this semantics can be reformulated for the simplicial set model as well

(Before this work it was not clear how to interpret such parametrised Higher Inductive Types even in a classical metatheory)

Quillen Model Structures

We represent equality proofs as *paths*, and a priori, we cannot expect to get in this way the required computational behavior of elimination rules of equality, as designed by Martin-Löf

However, Andrew Swan had an insight how to use notions from model structure to get a representation of Identity type with the expected computational behavior

For me, it was really unexpected; it showed the relevance of ideas from Quillen Model Structures for a purely computational/functional program question!

Quillen Model Structures

Christian Sattler could connect our generalised notion of open box to the notion of *cofibration*

He found around the same time (December 2015) that it is possible to define a Quillen Model Structure on all presheaves (not necessarily fibrant)

One crucial component in building this structure is the fact that we have a *fibrant* universe of fibrant presheaves

Variation of this has been formalised, by Simon Boulier (2018)

This is a *reverse* of the classical picture, where one starts from a Quillen Model Structure on simplicial sets to justify the model of univalence!

Quillen Model Structures

Steve Awodey has a thorough analysis of such "type theoretical" model structure, extracting 3 key properties

-*Frobenius* (and right properness: trivial cofibrations and equivalences are preserved by pullbacks along fibrations)

-*Equivalence Extension Property*

-*Fibration Extension Property* (a.k.a. "Joyal's trick")

Why is it interesting to have such a constructive understanding of this Quillen Model Struture?

Constructive models

(1) Dependent type theory with universes is much weaker than ZFC with universes (technically, even much weaker than Π^1_2-comprehension, cf. Martin-Löf's paper on *The Hilbert-Brouwer controversy resolved?*)

(2) Direct to have presehaf models and then sheaf models

Joyal had to use the technique of "Boolean localisation" (Barr's Theorem) in his 1984 letter to Grothendieck in order to define a Quillen Model Structure on simplicial sheaves

(3) "Direct" to have recursive models, cf. work of Uemura and then Swan and Uemura

Constructive models

We have constructive models of univalence, and Quillen Model Structures satisfying strong properties

Question

M. Shulman *The Derivator of Setoids*, 2021

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice?

Question

In particular, there are now at least two constructive homotopy theories - the aforementioned simplicial sets and the equivariant cartesian cubical sets of [ACC+21] - that can classically be shown to present the homotopy theory of spaces. However, it is not known whether they are constructively equivalent to each other. Thus one may naturally wonder: if they are not equivalent, which is the "correct" constructive homotopy theory of spaces? Or, perhaps, are they both "incorrect"? What does "correct" even mean?

We can give some elements from the point of view of constructive mathematics

Some troubling points

(1) Like for the groupoid model, one would expect countable choice to hold in these models, but this does not seem to be the case

(2) Related to the last point, one would expect that propositional truncation can be defined like in the setoid model, but this is not the case

Some troubling points

Some of these models validate the *negation* of excluded-middle (Sattler)!

Some troubling points

It is possible to define a modified form of Model Structure on simplicial sets *with* decidable degeneracies (work of Simon Henry) and this is related to a previous attempt of defining a semisimplicial model of type theory (work with Bruno Barras and Simon Huber)

However, we don't seem to get a model of dependent type theory in this way

We can interpret a weak form of dependent products, and it does not seem possible to "strictify" the model

A positive point

It is possible to define (pre)sheaf models of type theory (stacks)

In particular the notion of descent data can be seen to define a strict left exact modality (closely connected to the cobar operation used by Mike Shulman in his semantics in higher topos, but, in this setting, it defines a *lex modality*)

This is a *strict* pointed functor D, with a natural transformation $\eta_A : A \to DA$. acting also on families, and we extracted a condition for this to define a left exact modality

η_{DA} *and* $D\eta_A$ *should be path equal and should be equivalence*

By localisation, we get a (strict) model of dependent type theory with univalence and higher inductive types

A new model

I will now describe a new insight (due to Christian Sattler) which seems to provide a positive answer to Mike Shulman's question

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice?

We should be able to get both a model of dependent type theory with univalence and higher inductive types with a satisfactory corresponding Quillen model structure in a constructive metatheory

A "correct" model can be obtained by *localisation* of one of the cubical models

Δ and $\Box$

Δ is the category of finite nonempty linear posets

Δ_+ is the category of finite nonempty linear posets, with injective maps

$\Box$ is the category of finite nonempty posets

This is the dual of the category of nondegenerate f.p. distributive lattices

$$\Delta_+ \to \Delta \to \Box$$

Δ and $\Box$

We have the inclusion $\Delta_+ \to \Box$

This defines a strict monad D on the model $Ps(\Box)$

It is direct to check that this strict monad satisfies that η_{DA} and $D\eta_A$ are path equal and are equivalences

Hence we get by localisation a model of dependent type theory with univalence and higher inductive types

Δ and $\Box$

This model does not have any of the troubling points we listed above

(1) countable choice holds (even if it does not hold in the meta theory)

(2) propositional truncation can be defined as expected

(3) Whitehead principle holds: a map $f : X \to Y$ is an equivalence iff $\pi_0(f)$ bijection and all $\pi_n(f, x)$ are isomorphisms

Δ and $\Box$

We have the following

Let A be a type over Γ which is a family of propositions, with A which is D-modal, then

any section on points can be extended to a global section

Cubical type theory

All the syntactical operations (as long as we do not use reversal) are validated by this semantics

Classical compatibility with excluded-middle and axiom of choice

Δ and $\square$

This should provide a constructive explanation of homotopy types!

We obtain a Quillen Model Structure on $Ps(\square)$, which not only corresponds to spaces classically, but should also be well behaved constructively

For instance, we can define the nerve of a category by taking $N(C)(X)$ to be the set of functors from X to C and we expect Quillen's Theorems A and B to be (constructively) valid in this setting