Lorenzen and constructive mathematics

Introduction

The goal of this paper is to present a short survey of some works of Lorenzen in constructive mathematics, and its influence on recent development in mathematical logic and constructive algebra. We also present some work in measure theory which uses in an essential way Lorenzen's contributions.

1 Lorenzen as a mathematician and a logician

The school of mathematics in Germany between the two wars was truly exceptional (Noether, Herglotz, Artin, Schmidt, Krull, Hasse, ...). This is described in P. Roquette's survey [36], which emphasizes in particular the importance of the work of Hasse. Lorenzen was Hasse's student, and so was in direct contact with several members of this school.

A new feature was the use of highly non effective methods in *algebra*. The axiom of choice was used to show the existence of prime ideals (Krull), or to show the existence of real or algebraic closure of a given field. A striking example was the use of the real algebraic closure by Artin and Schreier to solve Hilbert's 17th problem [1].

The following extract of a letter from Krull to Scholz (1953) illustrates well how Lorenzen's contribution was perceived: *At working with the uncountable, in particular with the well-ordering theorem, I always had the feeling that one uses fictions there that need to be replaced some day by more reasonable concepts. But I was not getting upset over it, because I was convinced that at a careful application of the common "fictions" nothing false comes out, and because I was firmly counting on the man who would some day put all in order.* Lorenzen has now found according to my conviction the right way...

Lorenzen was quite unique in this group of mathematicians in being aware of works in logic, in particular the work of Gentzen. He was able to connect his work in algebra (lattice theory, Dedekind) with proof theory. While connections between lattice theory and logic were known since the work of Peirce and Schröder [34, 37], connections between lattice theory and *proof theory* were quite original (except for the previous work of Skolem [40]). On this topic, Lorenzen seems to be now mainly known only for the following result, which is actually only implicit in his fundamental paper [28].

Theorem 1.1 *A lattice is* distributive *if, and only if, it satisfies the (cut) rule*

$$\frac{a \wedge c \leqslant b \qquad a \leqslant b \vee c}{a \leqslant b}$$

This result is cited e.g. in Curry's book, Theorem B9, Chap. 4 [14]. As we shall see below, the connections between proof theory and lattice theory discovered by Lorenzen go much deeper than this result.

2 Lorenzen's analysis of Gentzen's work

2.1 Consistency proof

Gentzen's consistency proof by Lorenzen [28] is presented as a proof about an infinitary cut-free calculus showing that the cut rule is *admissible* ("zulässing"). Two highly original features of his argument are that the metatheory is *constructive* (with use of generalised inductive definitions) and that there is no ordinal analysis. At about the same time, and independently, P.S. Novikov had a similar analysis, and also introduced the notion of admissible/derivable rule [33, 10].

Apart from Novikov, most treatments in proof theory (Gentzen, Schütte, Takeuti) involve ordinal analysis. From a *constructive* point of view (and for me personally) the purely inductive presentation is much clearer. For strong calculus, such as Π^1_1-analysis, one can even argue that the ordinal analysis is a diversion. For instance, Takeuti proves consistency of this system with a system of ordinal diagrams in a finitary way. To have a *constructive* explanation of Π^1_1 comprehension, one needs further, as noticed e.g. in Kreisel's review of Takeuti's proof [21] or in Feferman's review of Takeuti's book on proof theory [15], to explain that ordinal diagrams are well-founded in an *intuitionistic* theory of inductive definitions. A direct explanation of Π^1_1-comprehension in an intuitionistic theory of inductive definitions (such as one obtained by use of Buchholz's Ω-rule [6]) seems thus to be preferred. Furthermore, the analysis of Lorenzen provides, as we shall see below, an effective description of the free σ-complete Boolean algebra on a given Boolean algebra.

To allow generalized inductive defined objects in a constructive setting was highly original. Apart from Novikov, the only example I could find are proofs in [31]. There, however, infinitary objects are not represented directly but only via coding as recursively enumberable sets (which arguably obscures the main ideas).

In the paper [26], Lorenzen and Myhill's paper analyses different ways to define subsets of natural numbers and introduce the following stratification:

1. By explicit definition, quantifying only over natural numbers.

2. By inductive definition, quantifying only over natural numbers.

3. By explicit definition, quantifying only over the (denumerable) totality of sets previously obtained.

4. By inductive definition, with the same restriction on quantifiers.

5. By uninhibited use of function-quantifiers.

Use of generalized inductive definitions (4) is presented as the "method of Lorenzen" exposed in [23], with the comment that this "exhausts those means of definition at present known which are acceptable from a standpoint which rejects the actual infinite". This analysis is quite similar to the one of Martin-Löf for instance in his paper [32].

The last method (5) is impredicativity which has no constructive justification

The method (4) goes beyond what has been called "predicative" mathematics, after the work of Schütte and Feferman [38, 16], but it is needed in constructive mathematics, as shown by Lorenzen in his analysis of Cantor-Bendixson Theorem (which is explained below).

2.2 Inversion principle

In Lorenzen's description of the mathematical universe, we have a calculus of inductively defined objects and inductive proofs/recursively defined functions on these objects.

For instance we describe inductively natural numbers by two production rules

$$\rightarrow \ | \qquad\qquad x \ \rightarrow \ x|$$

but we also describe inductively the relation of equality in the same way by the two production rules

$$\rightarrow \ | = | \qquad\qquad x = y \ \rightarrow \ x| = y|$$

One important discovery of Lorenzen is the *inversion principle* [23]: with this inductive description of equality, we have, as an *admissible* rule

$$| = x| \ \rightarrow \ \perp$$

since there is no way to derive an equality of the form $| = x|$.

This way of describing objects and proofs is now common practice in computer science. It is e.g. used extensively for expressing and proving properties of semantics of programming language (as in Kahn's natural semantics [20]) in interactive proof systems. Just to give an example, Lorenzen's paper [24] could almost be written as it is in proof systems for type theory.

In 1992, we noticed that this inversion principle corresponds to the notion of *pattern-matching* in functional programming [12]. This provides a convenient notation for inductive proofs, which is closely connected to the work [18] on definitional reflection. More recent works in this direction are N. Zeilberger's [41] and J. Cockx Ph.D. thesis [11].

2.3 Entailment relations

The paper [28] contains a deep application of proof theory to the study of distributive lattice, via the notion of *entailment relation*. An entailment relation is a relation $a_1, \dots, a_n \vdash b_1, \dots, b_m$ between finite subsets of a given abstract set such that

1. $X \vdash Y$ if X and Y intersect
2. $X \vdash Y$ if $X' \vdash Y'$ and $X' \subseteq X$ and $Y' \subseteq Y$
3. $X \vdash Y$ if $X, a \vdash Y$ and $X \vdash Y, a$

Entailment relation is the key notion for presenting distributive lattices/spectral spaces in an elegant way, as explained in [7]. If D is a (bounded) distributive lattice, an *interpretation* of $E, \vdash$ is a map $j : E \to D$ such that $X \vdash Y$ implies $\wedge j(X) \leqslant \vee j(Y)$. By universal algebra, there exists a universal interpretation $i : E \to L$: it is an interpretation such that, for any other interpretation $j : E \to D$ there is a *unique* map $f : L \to D$ such that $j = fi$. The following result [7] is essentially stated as such in [28].

Theorem 2.1 *Let $E, \vdash$ be an entailment relation. If L, $i : E \to L$ is the universal interpretation then we have $X \vdash Y$ if, and only if, $\wedge i(X) \leqslant \vee i(Y)$.*

Let us give an example in algebra. On a given domain R, a *valuation* for R is a domain $V \supseteq R$ in the field of fractions K of R such that, for any $a \neq 0$ in K, we have a in V or a^{-1} in V. A fundamental result, proved using Zorn's Lemma, is that an element of K is integral over R (i.e. root of a unitary polynomial in $R[X]$) if, and only if, it belongs to all valuation domain. Lorenzen was able to describe directly and effectively a relation $X \vdash Y$, which, classically, would be equivalent to the following relation: for all valuation domain V, if $X \subseteq V$ then V meets Y.

Lorenzen's description was the following [27]. If $x_1, \dots, x_n$ are elements in the fraction field of R we write $(x_1, \dots, x_n)$ the R-module generated by $x_1, \dots, x_n$.

Theorem 2.2 *The relation (for non zero element of the field of fractions of R)*

$$a_1, \dots, a_n \vdash b_1, \dots, b_m \quad \leftrightarrow \quad 1 \in \Sigma_{i>0}(a_1 b_1^{-1}, \dots, a_n b_m^{-1})^i$$

is an entailment relation, which is classically equivalent to the fact that if V is an arbitrary valuation domain and all elements a_i belongs to V then one of the element b_j is in V.

For instance $a \vdash b$ holds if, and only if, b is integral over a. In particular, b is integral over R if, and only if, we have $\vdash b$, which can be seen as a constructive version of the result that an element is integral if, and only if, it belongs to all valuation domain. We think this example illustrates well the way Lorenzen's work provides a constructive analysis of non effective methods in algebra (as evocated in Krull's letter cited above).

This was rediscovered in [13] but Lorenzen's analysis is more perspicious, relying on the following key Lemma, which shows that the cut-rule is valid for the relation defined in Theorem 2.2.

Lemma 2.3 *For any R-module I and $c \neq 0$ in K, if 1 in $I[c] = \Sigma_{n \geqslant 0} c^n I$ and in $I[c^{-1}] = \Sigma_{n \geqslant 0} c^{-n} I$ then 1 in I.*

Proof. (following [27]) It follows from the hypothesis that we have an equality of the form

$$(c^{-n}, \dots, c^{-1}, 1, c, \dots, c^n) = (c^{-n}, \dots, c^{-1}, 1, c, \dots, c^n)M$$

where M is a square matrix with coefficients in I. We then write $(c^{-n}, \dots, c^{-1}, 1, c, \dots, c^n)(1 - M) = 0$ and hence the determinant of $1 - M$ is 0, which shows that 1 is in I. □

3 Proof theoretic analysis of point-free spaces

In this subsection, we want to present Lorenzen's analysis of Cantor-Bendixson's Theorem [25]. This analysis was crucially needed in Kreisel's work [22]. What is remarkable about this result is that, as shown by Kreisel, Cantor-Bendixson's Theorem requires methods going beyond what has been called "predicative mathematics" by Feferman and Schütte [15, 38].

In order to present this analysis as simply as possible, we will do it for Cantor space instead of $[0,1]$ (as is done in [25]). As a set of points, the Cantor space is the set Ω of infinite binary sequences $\omega = \omega_0, \omega_1, \omega_2, \dots$ As a point-free space, where we describe directly in algebraic (and effective) term its compact open subsets, it can be seen as the Boolean algebra of propositional logic C, i.e. the Boolean algebra freely generated by countably many formal atoms written $\omega_k = 1$ (of formal complement $\omega_k = 0$) For instance $\omega_1 = 0 \wedge \omega_3 = 1$ represents a compact open subset of Ω, namely all sequences ω such that $\omega_1 = 0$ and $\omega_3 = 1$.

More generally any compact totally disconnected space X can be described using its associated Boolean algebra B of compact open subsets [19]. An *open* subset of X corresponds to an *ideal* of B, and so (by taking the complement) there is a correspondance between closed subsets of X and ideals of B. If x is an element of B, it defines the ideal $\downarrow x = \{y \in B \mid y \leqslant x\}$.

The key insight of Lorenzen was that the derivative of a closed subset (the set of non isolated points) gets a more effective description via this correspondance. It relies on the following simple observation.

Lemma 3.1 *The open subset of* isolated *points of X corresponds to the ideal of B of elements x such that $\downarrow x$ is finite.*

It follows from this that the *kernel* of F, which is obtained by iterating the derivative operation transfinitely (taking the intersection at limit ordinals) can be described *effectively* by a *generalized* inductive definition for Cantor space (and for $[0,1]$).

For Cantor space, we can describe more concretely an open subset (or closed subset as a complement) as a set of binary finite sequences U which is upper closed and such that s is in U if both $s0$ and $s1$ are in U. For each n we have 2^n elements $s_{i_0 \dots i_{n-1}} = [\omega_0 = i_0] \wedge \dots \wedge [\omega_{n-1} = i_{n-1}]$ of length n. Starting from a set U, which describes the complement of a given closed subset F of Cantor space, let us then consider the following *generalised* inductive definition, which describes a set S of binary sequences

1. s is in S if s is in U
2. s of length k is in S if there exists a *fixed* l such that, for each n big enough, at least $2^{n-k} - l$ elements $s_{i_0 \dots i_{n-1}}$ extending s are in S

This set S defines an open subset of Cantor space.

Theorem 3.2 *The complement of the open subset corresponding of S is the* kernel *of F.*

4 Measure theory

4.1 Borel subsets of Cantor space

The analysis by Lorenzen of Gentzen's cut-elimination contains an effective description the σ-complete Boolean algebra generated by a given Boolean algebra. More generally, given an entailment relation $E, \vdash$ as defined above, Lorenzen describes the σ-complete Boolean algebra B with an interpretation $v : E \to B$ universal for this property. He then shows

Theorem 4.1 *For the universal σ-complete Boolean algebra B with an interpretation $v : E \to B$, we have*

$$a_1, \dots, a_n \vdash b_1, \dots, b_m \quad \leftrightarrow \quad v(a_1) \wedge \dots \wedge v(a_n) \leqslant v(b_1) \vee \dots \vee v(b_m)$$

This result is cited in the reference [3] (which might be surprisingly the only published reference to the fundamental paper [28]). If we start from the Boolean algebra C of *propositional logic* which is the Boolean algebra generated from countably many atoms we get a σ-complete Boolean algebra B.

As explained above, C can be seen as a point-free presentation of *Cantor space*, which is the set Ω of all infinite binary sequences $\omega = \omega_0, \omega_1, \dots$ What is remarkable is that B can be seen as a point-free presentation of the σ-complete Boolean algebra of *Borel sets* on Cantor space. This was noticed by P. Martin-Löf [31]. If we start from the Boolean algebra with two elements we get the σ-complete Boolean algebra of *hyperarithmetical propositions.*

In this point-free view, a Borel set X is given inductively: X is a propositional formula or X is of the form $\bigvee_n X_n$ or X is of the form $\bigwedge_n X_n$. Lorenzen defines a sequent calculus $X_1, \dots, X_n \vdash Y_1, \dots, Y_m$ and proves that the cut-rule is admissible. The same analysis is done in [31].

We can define $X \subseteq Y$ by $X \vdash Y$. We have $X \subseteq X$ by induction on X and, using cut-elimination, $X \subseteq Z$ if $X \subseteq Y$ and $Y \subseteq Z$

An example of a point-free description is the set of *normal* binary sequences

$$N = \bigwedge_k \bigvee_m \bigwedge_{n \geqslant m} b_{n,k}$$

with $b_{n,k}$ a point-free representation of

$$\{ \ \omega \in \Omega \mid -\frac{1}{k} \leqslant \frac{\Sigma_{i<n}(2\omega_i - 1)}{n} \leqslant \frac{1}{k} \ \}$$

In the classical approach this is thought of as a set of points (the complement of which is not countable and of measure 0). In the present setting, it is a purely symbolic expression. We can test an effective theory of measure of Borel sets by the fact that this set, defined in this "symbolic" way, should be of measure 1.

4.2 Borel's measure problem

As explained above, Borel sets can be described inductively. The following is then a natural question: can we define the measure $\mu(X)$ of a Borel set X by induction on X? Borel's own formulation [5] was the following (for subsets of $[0,1]$): we design a formal theory which describes how the measure should work, and we have to prove that this formal theory is *consistent.*

As presented by Lusin [30], it can be seen as a *coherence problem*: we have to provide an inductive definition of the measure $\mu(X)$ of a Borel set X such that $X \vdash Y \ \rightarrow \ \mu(X) \leqslant \mu(Y)$. Lusin in his book [30] asked for a purely inductive solution of this problem, and called this question *Borel's measure problem.* The usual definition of measure (Lebesgue, Daniell, Bourbaki) shows the consistency of the theory of the measure, but it does it in a non effective way and goes *beyond* inductive reasonings.

4.3 An inductive solution of Borel's measure problem

Here I explain how to define recursively $r < \mu(X)$ as a *hyperarithmetical* proposition by induction on X. We take the usual measure on Cantor space: if X is a propositional formula $\mu(X)$ is a rational and $r < \mu(X)$ is 0 or 1. For instance $\mu(\omega_1 = 0 \wedge \omega_3 = 1) = 1/4$.

As stated by Borel, it is reasonable, starting from this closed open subsets, to require that the measure of a *disjoint* countable union is the sum of the measure (if this sum actually converges). Thus, if we define $X_{n+1} = \wedge_{i<n}[\omega_i = 0] \wedge [\omega_n = 1]$ and $X_0 = \bigwedge_k [\omega_k = 0]$, we have $1 = \bigvee_n X_n$ and $\mu(1) = 1$ and $\mu(X_0) = 0$ and $\mu(X_{n+1}) = 1/2^{n+1}$. We can then check consistency since we have $1 = 0 + 1/2 + 1/4 + \dots$.

The main difficulty in this inductive approach is: how to define $r < \mu(X)$ if X is a disjunction or conjunction? One solution is provided by the remarkable paper of F. Riesz [35]: we instead define recursively $r < \mu(b \wedge X)$ for *each* propositional formula b. We introduce a new relation $r < \mu(X, b)$ which represents $r < \mu(b \wedge X)$, which can be defined inductively on X and we recover $r < \mu(X)$ as $r < \mu(X, 1)$.

The insight of F. Riesz was that, if $X = \bigvee_n X_n$ then

$$\mu(b \wedge X) = \bigvee_{\substack{b = b_1, \dots, b_k \\ n_1 < \dots < n_k}} \mu(b_1 \wedge X_{n_1}) + \dots + \mu(b_k \wedge X_{n_k})$$

where $b = b_1, \dots, b_k$ is a *partition* of b.

So $\mu(b \wedge X)$ is defined in term of $\mu(c \wedge X_n)$ for some $c \leqslant b$!
If $X = c$ then we can compute $r < \mu(b \wedge c)$ and this is the value of $r < \mu(X, b)$.
If $X = \bigvee_n X_n$ then $r < \mu(X, b)$ is the formula

$$\bigvee_{\substack{b=b_1,\dots,b_k \\ r=r_1+\cdots+r_k \\ n_1<\cdots<n_k}} r_1 < \mu(X_{n_1}, b_1) \ \wedge \ \dots \ \wedge \ r_k < \mu(X_{n_k}, b_k)$$

For $X = \bigwedge_n X_n$ we should have $\mu(b \wedge X) = \mu(b) - \mu(b \wedge \bigvee_n X'_n)$ and

$$\mu(b \wedge \bigvee_n X'_n) = \bigvee_{\substack{b=b_1,\dots,b_k \\ n_1<\cdots<n_k}} \mu(b_1 \wedge X'_{n_1}) + \cdots + \mu(b_k \wedge X'_{n_k})$$

From this, we deduce the value of $r < \mu(X, b)$, as the formula

$$\bigvee_{r<s} \bigwedge_{\substack{b=b_1,\dots,b_k \\ n_1<\cdots<n_k}} \bigvee_{s=s_1+\cdots+s_k} s_1 < \mu(X_{n_1}, b_1) \ \wedge \ \dots \ \wedge \ s_k < \mu(X_{n_k}, b_k)$$

In this way, we define recursively $r < \mu(X, b)$ as a hyperarithmetical formula.
It is then possible [9] to show *purely inductively* the following result.

Theorem 4.2 *If we have $X \vdash Y$ then $[r < \mu(X, b)] \leqslant [r < \mu(Y, b)]$. Hence if X and Y defines the same Borel subset of Cantor space, we have $\mu(X) = \mu(Y)$.*

This shows the consistency of our definition: if X and Y represent the *same* Borel set then $r < \mu(X, b)$ and $r < \mu(Y, b)$ are equal.

As an application, we can show, purely inductively, that $r < \mu(N, 1)$ is provable for each $r < 1$, where N is the symbolic representation of the set of normal binary sequences described above. We get in this way a proof of $\mu(N) = 1$ which only involves inductive reasoning.

5 Game semantics

In this last section, I only briefly mention the work on game semantics, interpreting a proof as a *winning strategy*. In particular, Lorenzen has a suggestive analysis (e.g. in [29]) of the formula $\neg\neg a \rightarrow a$ and why it is not intuitionistically valid. The idea is to consider a statement a for which the opponent has a proof, which is not known by the proponent. If the opponent asserts $\neg\neg a$, the proponent (who does not know the proof of a) has to challenge the opponent by asserting $\neg a$ (hoping that the opponent does not know the proof of a either). But then the opponent wins by giving the proof of a.

This idea of game interpretation has been refined in various ways. An extension of this interpretation to *analysis* is described in [4], providing in particular a different interpretation than Spector (1961). See also the work [17] interpreting of the axiom of determinacy.

I suggested an analysis of *cut-elimination* based on this interpretation, describing cut-elimination as an interaction between two strategies that both can backtrack [13]. We can in this way give a proof of termination of the cut-elimination process, essentially different from Gentzen's analysis. This has recently been used by F. Aschieri (2015) for proving a non trivial refinement of Gentzen's upper bound (with a tower of exponential) in term of the level of *backtracking* of the strategies.

For instance, if *one* strategy has only one level of backtracking then we have a single exponential (whatever the complexity of the cut formula).

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