$x^2 + 1$ IS POSITIVE

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Abstract. We present an elementary proof of a lemma in Krivine [2]. As a corollary, we get an elementary constructive proof of Kadison-Dubois' theorem.

Introduction

The goal of this note is to present an alternative, and we think simpler, proof of the following remark in [2]. Let R be a commutative Q-algebra with a subset Ω such that all rationnals $q \geq 0$ are in Ω and if $x, y \in \Omega$ then $x + y, xy \in \Omega$. We assume furthermore that for any $x \in R$ there exists n such that $n - x \in \Omega$. Under these conditions, we have $x^2 + 1 \in \Omega$ and hence $x^2 + r \in \Omega$ for all $x \in R$ and all rational $r > 0$.

Following Krivine's proof [2, 3], but formulated in a point-free way using the techniques of [1], we get a constructive elementary proof of Kadison-Dubois.

A Simple Lemma

Our observation is that the following special case of a theorem in [2] is enough to conclude $x^2 + 1 \in \Omega$ and has a direct elementary proof.

Lemma 0.1. *For any n we can write $x^2+1 = P(n-x, n+x)$ where $P(X, Y)$ is a rational polynomial with coefficients ≥ 0.*

Proof. We show that we can find $P(u, v)$ homogeneous. We use the change of variables $y(n + x) = n - x$. The question reduces to find k such that all coefficients of

$$(1 + y)^k(1 - 2uy + y^2)$$

are ≥ 0 where $u(n^2 + 1) = n^2 - 1$. It is direct and elementary that this is the case as soon as $n^2 - 1 \leq k$. If we write

$$\Sigma a_i y^i = (1 + y)^k(1 - 2uy + y^2)$$

we can take $P(u, v) = \Sigma a_i u^i v^{k+2-i}$. □

Notice that $P(u, v)$ is of degree $n^2 + 1$. We don't know if this degree is optimal.

Corollary 0.2. *If R is a Q-algebra with an archimedian order containing Q^+, but without assuming that all square are ≥ 0 then we have $x^2 + r \geq 0$ for all $x \in R$ and all rationals $r > 0$.*

Let R be a Q-algebra with an archimedian order containing Q^+, but without assuming that all square are ≥ 0. This means that we have a subset $\Omega \subseteq R$ closed under addition and multiplication, $Q^+ \subseteq \Omega$ and for any $x \in R$ there exists n such that $n - x \in \Omega$. Let now Ω' be the *cône* generated by Ω that is the least subset of R closed under addition and multiplication, containing Ω and all square $x^2,\ x \in R$.

Corollary 0.3. *If $x \in \Omega'$ and r is a rational > 0 then $x + r \in \Omega$.*

Proof. It is enough to notice that the set of all elements $x \in R$ such that $x + r \in \Omega$ for all $r > 0$ is a cône containing Ω and Q^+. □

Using the techniques of [1] we get a constructive proof of the following result.

Theorem 0.4. *Let $Q(x_1, \dots, x_n)$ a rational polynomial which is $\geq r$ on $[0, 1]^n$, r rational > 0 then we can write $Q = P(x_1, \dots, x_n, 1-x_1, \dots, 1-x_n)$ where P is a rational polynomial with all coefficients ≥ 0.*

An application is for spectral analysis: if $A_1, \dots, A_n$ is a family of auto-adjoint positive operators which commute pairwise, then we can use this theorem to define $f(A_1, \dots, A_n)$ for any continuous functions on $\Pi_i[0, ||A_i||]$.

References

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