A Constructive Analysis of the Stone-Weierstrass Theorem

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Introduction

This note can be seen as a constructive comment on an older note of M.H. Stone [StII]. We give a simple, constructive localic proof of the Stone-Weierstrass theorem and a constructive treatment of Gelfand duality in the real case, using only elementary algebraic arguments. This gives also a particularly simple formal presentation of compact Hausdorff spaces X (and the associated spaces of probability measures).

1 A propositional theory

Let E be a lattice ordered abelian group (or *l-group*). The elements of E are written $a, b, c, \dots$ The operation of E is written additively and the sup operation is written $a \vee b$. The following assertions are standard, and have elementary, though not obvious, proofs [Birk, Lux].

Proposition 1.1 For $n \geq 1$ we have $n(a \vee b) = na \vee nb$ and $n(a \wedge b) = na \wedge nb$, also $na \geq 0$ implies $a \geq 0$.

This implies that E can be embedded as an l-group in a *divisible* lattice ordered group where for each x and $n \geq 1$ there exists exactly one solution for $ny = x$. To simplify the presentation, we will assume in the following that E is divisible; it has then naturally the structure of a Riesz space over the set of rationals Q [Lux].

We write as usual a^+ for $a \vee 0$ and a^- for $(-a) \vee 0$. We can define $a \wedge b$ as $a + b - a \vee b$. We say that a is *positive* iff $a \geq 0$. Let P be the set of positive elements. We write $a \perp b$ if $a, b \in P$ and $a \wedge b = 0$. We shall also use the following standard results [Birk].

Proposition 1.2 We have $a = a^+ - a^-$ and $a^+ \perp a^-$. Also, for positive a, b, c, if $a \perp b$ and $a \perp c$ then $a \perp (b + c)$.

If $b \in P$ we write $a \preceq b$ to mean that there exists $n \geq 1$ such that $a \leq nb$. If $a, b \in P$ we write $a \sim b$ iff $a \preceq b$ and $b \preceq a$. The following remark will be important, and it has a direct proof.

Proposition 1.3 For $a, b, c \in P$ we have $a \preceq b$ if $a \leq b$. Also

$$a \preceq c, b \preceq c \to a \vee b \preceq c \qquad c \preceq a, c \preceq b \to c \preceq a \wedge b,$$

hence the structure $(P/\sim, \wedge, \vee, \preceq)$ forms a distributive lattice L.

We associate to E a coherent propositional theory T [Joh, MP]. The atomic propositions are the elements of E and the axioms are

1. $a, -a \vdash$
2. $a \vdash$ if $a \leq 0$
3. $a + b \vdash a, b$
4. $a \vee b \vdash a, b$

Here, following Gentzen, we use , as meaning conjunction when appearing to the left and disjunction when appearing to the right of the $\vdash$ symbol.

Proposition 1.4 In T we have $a \vdash b$ whenever $a \leq b$. In particular $a \vdash a^+$. We also can prove

$$a, b \vdash a + b \qquad a, b \vdash a \wedge b \qquad a^+ \vdash a$$

Proof. We have $a \vdash a + b, -b$ and $-b, b \vdash$. By cut, we get $a, b \vdash a + b$.

If $a \leq b$ we have $a - b \leq 0$ and hence $a - b \vdash$. Since $a \vdash a - b, b$ we get by cut $a \vdash b$.

Since $0 \vdash$ and $a \vee 0 \vdash a, 0$ we get by cut $a \vee 0 \vdash a$, that is $a^+ \vdash 0$. It follows that we have

$$(a - b)^+, (b - a)^+ \vdash$$

Since $a = a \wedge b + (a - b)^+$ and $b = a \wedge b + (b - a)^+$ we have also

$$a \vdash a \wedge b, (a - b)^+ \qquad b \vdash a \wedge b, (b - a)^+$$

and hence by cut $a, b \vdash a \wedge b$. □

Theorem 1.5 The map $m : a \longmapsto a^+,\ E \to L$ is an interpretation of T in the distributive lattice L. Furthermore it is the *free* such interpretation. In particular, $a \vdash b$ is derivable in T iff $a^+ \preceq b^+$.

Proof. Using proposition 1.4, it is easy to see that if $a_1^+ \wedge \ldots \wedge a_n^+ \preceq b_1^+ \vee \ldots b_m^+$ then

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$$

is derivable in T. Also, it is direct that m is an interpretation of T. □

2 Stone-Weierstrass Theorem

We assume now that E has a strong unit 1: we have $0 \leq 1$ and $a \preceq 1$ for all $a \in E$. An important consequence [Birk] is that if $a \in P$ and $a \wedge 1 = 0$ then $a = 0$. We notice that in this case, E has naturally the structure of a seminormed vector space [MP] by defining $a \in N(q)$ as $-q'1 \leq a \leq q'1$ for some $q' < q$. We can verify rather directly that this defines a seminormed structure on E.

We let T_1 be the theory T extended with the axiom $\vdash 1$. To this theory T_1 we can associate canonically a spectral space $Sp(T_1)$, and we let $Frm(T_1)$ be its frame of open [Joh].

We associate now to E the following geometric propositional theory [MP] $S(E)$. The atomic propositions are, as before, the elements of E and the axioms are

1. $a, -a \vdash$

2. $a \vdash$ if $a \leq 0$

3. $a + b \vdash a, b$

4. $a \vee b \vdash a, b$

5. $\vdash 1$

6. $a \vdash \bigvee_{q>0}(a - q1)$

Had we not assume E to be divisible, the last axioms should have be formulated as $a \vdash \bigvee_n(na - 1)$.

Theorem 2.1 The theory $S(E)$ describes a compact regular locale X. The points of X can be seen as models of the theory $S(E)$ and can be identified classically with positive linear mappings $\phi : E \to R$ preserving sups.

Proof. Any element a of E can be seen as a basic open $D(a)$ of the space X. Notice that, because of the axiom $a \vee b \vdash a, b$, we have $D(a \vee b) = D(a) \cup D(b)$. We also have directly $D(qa) = D(a)$ for $q > 0$.

We have an interpretation $m : E \to Frm(T_1)$ of the theory $S(E)$ in the spectral frame $Frm(T_1)$ given by $m(a) = \bigvee_{q>0}(a - q1)$. Indeed, it is directly checked that all the axioms of $S(E)$ are valid in this interpretation. It follows from this remark that X is a retract of $Sp(T_1)$, and hence that X is stably compact [Joh]. Since we have

$$a - q1, -a + q1 \vdash \qquad \vdash a, -a + q1$$

each open $D(a - q1)$, $q > 0$, is well-inside the open $D(a)$ and X is compact regular [Joh].

A point α of the space X is a model of the theory $S(E)$ [MP]. We can associate to α a map $\phi_\alpha : E \to R$ by defining $\phi_\alpha(a)$ to be the supremum of $q \in Q$ such that $a - q1$ holds in the model α. This is well-defined because 1 is a strong unit, and this is a positive linear functional preserving binary sups. Conversely any such positive linear function $\phi : E \to R$ defines a model of $S(E)$ by interpreting a as $0 < \phi(a)$. $\square$

Theorem 2.2 If $X = D(a)$ then $1 \preceq a$ in E

Proof. If $\vdash a$ in $S(E)$ we have $1 = \bigvee_{q>0} a - q1$ in $Frm(T_1)$ and hence $1 \vdash a - q1$ is derivable in T_1 for some $q > 0$. By theorem 1.5 this implies $1 \preceq (a - q1)^+$. It follows from this that $1 \perp (a - q1)^-$, since $(a - q1)^+ \perp (a - q1)^-$ and hence that $(a - q1)^- = 0$. Hence $a - q1 = (a - q1)^+$ is positive and $1 \preceq a$. $\square$

Each element of a defines a formal continuous map $\hat{a} \in C(X)$

$$\hat{a} : \alpha \longmapsto \phi_\alpha(a)$$

The last theorem can be seen as expressing that if $\hat{a}$ is a strictly positive function, then $\epsilon 1 \leq a$ in E for some rational $\epsilon > 0$. Another formulation of this result shows the analogy with Gelfand duality.

Corollary 2.3 We have $a \in N(q)$ iff $|\hat{a}(x)| < q$ for any $x \in X$.

It follows from this also that $\hat{a} = 0$ iff a is infinitesimal: we have $-1/n \leq a \leq 1/n$ for all n. The next proposition states the existence of partition of unity. A direct corollary can be seen as a constructive version of the Stone-Weierstrass theorem.

Proposition 2.4 If U_j is an arbitrary covering of X it is possible to find a partition of unity $p_1, \dots, p_n$ with $p_i \in [0,1]$ and $\Sigma p_i = 1$ and each open $D(p_i)$ is a formal subset of some U_j.

Proof. Given any covering U_j we can find positive elements $a_1, \dots, a_n$ such that the formal open $D(a_i)$ is a formal subset of some U_j and

$$X = D(a_1) \cup \dots \cup D(a_n) = D(a_1 \vee \dots \vee a_n)$$

This means that we have in $Sp(E)$

$$\vdash a_1 \vee \dots \vee a_n$$

and hence, by theorem 2.2, we have

$$1 \leq N(a_1 \vee \dots \vee a_n) = Na_1 \vee \dots \vee Na_n$$

for some $N \geq 1$. If we define $q_i = 1 \wedge Na_i \in [0,1]$ we have thus $\vee q_i = 1$. If we define next $p_i = q_i - (q_i \wedge \vee_{j<i} q_j)$, we have $p_i \in [0,1]$, each basic open $D(p_i) \subseteq D(a_i)$ is a formal subset of some U_j and $\Sigma_{j<i} p_j = \vee_{j<i} q_j$. In particular $\Sigma p_i = \vee q_i = 1$. $\square$

Corollary 2.5 If $f \in C(X)$ and $\epsilon > 0$ there exists $a \in E$ such that $|f(x) - \hat{a}(x)| < \epsilon$ for all $x \in X$.

Proof. We can find a family U_i, q_i such that U_i covers X and $|f(x) - q_i| < \epsilon$ for $x \in U_i$. If we take a corresponding partition of unity $p_i \in E$ and define $a = \Sigma q_i p_i$ we have

$$f(x) = \Sigma \hat{p}_i(x) f(x)$$

and hence

$$|f(x) - \hat{a}(x)| \leq \Sigma \hat{p}_i(x) |f(x) - q_i|$$

for all $x \in X$. Since $\hat{p}_i(x) > 0$ implies $x \in U_i$ it implies also $|f(x) - q_i| < \epsilon$ and hence $|f(x) - \hat{a}(x)| < \epsilon$ for all $x \in X$. $\square$

This corollary states that the functions $\hat{a},\ a \in E$ are dense in $C(X)$ and is analogous to the Stone-Weierstrass theorem.

If we don't assume divisibility in E, only the last statement has to be changed. We have to state that for any $\epsilon > 0$ and $f \in C(X)$ there exists $n \geq 1$ and $a \in E$ such that

$$|nf(x) - \hat{a}(x)| < n\epsilon$$

for all $x \in X$.

3 Space of measures

Similarly, we can consider the following geometric propositional theory

1. $a, -a \vdash$
2. $a \vdash$ if $a \leq 0$
3. $a + b \vdash a, b$
4. $\vdash 1$
5. $a \vdash \bigvee_{q>0}(a - q1)$

where we simply take away the axioms $a \vee b \vdash a, b$. One can show, in a similar way as we did for the space X, that this defines a compact regular space $M(X)$.

The space $M(X)$ can then be seen as the space of probability measure over X. Indeed a point of $M(X)$ defines a positive linear functional $I : E \to R$ such that $I(1) = 1$ which extends in a unique way to a positive linear functional $I : C(X) \to R$.

Compared to the usual expression of Stone-Weiestrass theorem, we have replaced the ring structure of $C(X)$ by its lattice structure. In another work [C1] we comment on another note of M.H. Stone [StI], where the ring structure is analysed.

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