Invariant Measure on Compact Space

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Introduction

Let X be a compact space, and G a commutative group acting on X. It is well-known (Kakutani's theorem) that there exists a G-invariant probablity measure on X. The formal interpretation of this statement gives a non trivial example of an entailment relation.

1 Measures as Orders

We consider first only measure on distributive lattices. To any distributive lattice L we associate an ordered commutative group $C(L)$. The elements are formal sums $\Sigma n_i x_i,\ x_i \in L$ and the order is defined by $\Sigma x_i \leq \Sigma y_j$ iff

$$\vee_I x_I \leq \vee_J y_I$$

where I ranges over all subsets of $[0, n-1]$ of cardinal k and where x_I denotes $\Pi_{i\in I} x_i$ and similarly for y_J.

Now a measure on X is a special kind of total order on $C(X)$ namely a total order that satisfies: if $p < x$ then there exists $p' > p$ such that $p' < x$. So we represent a measure on X as a total order on $C(X)$ that extends the given order $\leq$.

2 Entailment relations for Orders

We have a simple notion of entailment relations [Avr91, CC99, Sco71, Sco73, Sco74] to represent such order on $C(X)$: we define $X \vdash Y$ to mean that a non empty sum of elements in X is $\leq$ a sum of elements of Y. It is then direct that this defines an entailment relation, which is furthermore the least entailment relation such that

- $x \vdash y$ if $x \leq y$,
- $x + y \vdash x, y$,
- $x, -x \vdash$.

3 Invariant Entailment

Suppose now that G is a commutative group acting on X. We can analyse the space of G-invariant measures and show that it is not trivial. It turns out that this is a direct extension of the previous case. The group G acts now on $C(X)$. We want to characterise the least entailment relation such that

- $\vdash 1$,
- $x \vdash gx$ for $g \in G$,
- $x \vdash y$ if $x \leq y$,
- $x + y \vdash x, y$,
- $x, -x \vdash$.

It is enough to change the definition of $X \vdash Y$ to: a non empty sum of elements in $G(X \cup \{1\})$ is $\leq$ a sum of elements of GY. Indeed, let us show that if $X, x \vdash Y$ and $X \vdash x, Y$ then we have $X \vdash Y$. We have two relations of the form

$$a + \Sigma g_i x \leq b$$

$$c \leq \Sigma h_j x + d$$

where a, c are sums of elements on $G(X \cup \{1\})$ and b, d sums of elements in GY. But using the commutativity of G these two relations imply

$$\Sigma h_j a + \Sigma g_i c \leq \Sigma h_j b + \Sigma g_i d$$

and hence $X \vdash Y$ as desired.

This can be seen as a constructive formulation of Kakutani's fixed-point theorem. However our formulation is more general and applies to *any* ordered commutative group on which G acts.

References

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