HITs in cubical sets

Spheres, syntactical presentation

We define the circle S^1 by the rules

$$\frac{\Gamma \vdash}{\Gamma \vdash S^1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{base} : S^1} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{loop}\ r : S^1}$$

with the equalities $\mathsf{loop}\ 0 = \mathsf{loop}\ 1 = \mathsf{base}$.

Since we want to represent the *free* type with one base point and a loop, we add composition as a *constructor* operation hcomp^i (which bounds i in u)

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : S^1 \qquad \Gamma \vdash u_0 : S^1[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : S^1[\varphi \mapsto u(i1)]}$$

Given a dependent type $x : S^1 \vdash A$ and $a : A(x/\mathsf{base})$ and $l : \mathsf{Path}^i\ A(x/\mathsf{loop}\ i)\ a\ a$ we can define a function $g : \Pi(x : S^1)A$ by the equations[1]

$$g\ \mathsf{base} = a \qquad g\ (\mathsf{loop}\ r) = l\ r$$

This definition is non ambiguous since $l\ 0 = l\ 1 = a$ and we get *judgemental* computation rules. Finally

$$g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) = \mathsf{comp}^i\ A(x/v)\ [\varphi \mapsto g\ u]\ (g\ u_0)$$

where $v = \mathsf{fill}^i\ S^1\ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^j\ [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\ u_0$.

We have a similar definition for S^n taking as constructors base and $\mathsf{loop}\ r_1\ \dots\ r_n$.

Spheres, semantical presentation

We suppose to have a fresh name function on the set of names, with $\mathsf{fresh}(I)$ being a name not in I, and we write $I^+ = I, \mathsf{fresh}(I)$. We can define in a functorial way $f^+ : J^+ \to I^+$ extending $f : J \to I$ by sending $\mathsf{fresh}(I)$ to $\mathsf{fresh}(J)$. We also have for natural transformations the projection $p : I^+ \to I$ and the map $0 : I \to I^+$ (resp. $1 : I \to I^+$) sending $\mathsf{fresh}(I)$ to 0 (resp. 1).

A cubical set X is defined to be a family of sets $X(I)$ with restriction maps $X(I) \to X(J),\ u \longmapsto uf$ for $f : J \to I$ such that $u1_I = u$ and $(uf)g = u(fg)$ if $g : K \to J$.

We define first a cubical set $X(I)$ which is an "upper approximation" of the circle. An element of $X(I)$ is of the form base or $\mathsf{loop}\ r$ with $r \neq 0, 1$ in $\mathbb{I}(I)$ or of the form $\mathsf{hcomp}\ [\psi \mapsto u]\ u_0$ with $\psi \neq 1$ in $\mathbb{F}(I)$ and u_0 in $X(I)$ and u a family of elements u_f in $X(J^+)$ for $f : J \to I$ such that $\psi f = 1$. In this way an element of $X(I)$ can be seen as a well-founded tree. We can define uf in $X(J)$ for $f : J \to I$ by induction on u. We take $\mathsf{base}\ f = \mathsf{base}$ and $(\mathsf{loop}\ r)f = \mathsf{loop}\ (rf)$ if $rf \neq 0, 1$ and $(\mathsf{loop}\ r)f = \mathsf{base}$ if rf is 0 or 1. Finally $(\mathsf{hcomp}\ [\psi \mapsto u]\ u_0)f$ if $u_f 1$ if $\psi f = 1$ and $\mathsf{hcomp}\ [\psi f \mapsto uf^+]\ (u_0 f)$ if $\psi f \neq 1$ where uf^+ is the family $(uf^+)_g = u_{fg}$ for $g : K \to J$. This defines a cubical set.

We then define the subset $S^1(I) \subseteq X(I)$ by taking the elements base and $\mathsf{loop}\ r$ and $\mathsf{hcomp}\ [\psi \mapsto u]\ u_0$ such that u_0 in $S^1(I)$ and each u_f in $S^1(J^+)$ and $u_0 f = u_f 0$ and $u_f g^+ = u_{fg}$ for $f : J \to I$ and $g : K \to J$. This defines the sub-cubical set S^1 of X.

[1] For the equation $g\ (\mathsf{loop}\ r) = l\ r$, it may be that l and r are dependent on the same name i, and this could not work without a diagonal operation on names.

Propositional truncation, syntactical presentation

We define the propositional truncation $\|A\|$ of a type A by the rules:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \|A\|} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{inc}\ a : \|A\|} \qquad \frac{\Gamma \vdash u_0 : \|A\| \quad \Gamma \vdash u_1 : \|A\| \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{squash}\ u_0\ u_1\ r : \|A\|}$$

with the equalities $\mathsf{squash}\ u_0\ u_1\ 0 = u_0$ and $\mathsf{squash}\ u_0\ u_1\ 1 = u_1$.

As before, we add composition as a constructor, but only in the form[2]

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \|A\| \qquad \Gamma \vdash u_0 : \|A\|\,[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \|A\|\,[\varphi \mapsto u(i1)]}$$

This provides only a definition of $\mathsf{comp}^i\ \ \|A\|\ \ [\varphi \mapsto u]\ u_0$ in the case where A is independent of i, and we have to explain how to define the general case.

Given $x : \|A\| \vdash B$ and $q : \Pi(x_0 : \|A\|)(y_0 : B(x_0))(x_1 : \|A\|)(y_1 : B(x_1))\mathsf{Path}^i\ B(\mathsf{squash}\ x_0\ x_1\ i)\ y_0\ y_1$ and $f : \Pi(x : A)B(\mathsf{inc}\ x)$ we define $g : \Pi(x : \|A\|)B$ by the equations

$$\begin{array}{lcl} g\ (\mathsf{inc}\ a) & = & f\ a \\ g\ (\mathsf{squash}\ u_0\ u_1\ r) & = & q\ \ u_0\ (g\ u_0)\ u_1\ (g\ u_1)\ r \\ g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) & = & \mathsf{comp}^i\ B(v)\ [\varphi \mapsto g\ u]\ (g\ u_0) \end{array}$$

where $v = \mathsf{hcomp}^j\ [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\ u_0$.

Flattening an open box

We still have to define the general composition operation. We define first

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash u : \|A(i/r)\|}{\Gamma \vdash \mathsf{forward}\ r\ u : \|A(i/1)\|\,[(r = 1) \mapsto u]}$$

by the equations

$$\begin{array}{lcl} \mathsf{forward}\ r\ (\mathsf{inc}\ a) & = & \mathsf{inc}\ (\mathsf{comp}^i\ A(i \vee r)\ [(r = 1) \mapsto a]\ a) \\ \mathsf{forward}\ r\ (\mathsf{squash}\ u_0\ u_1\ s) & = & \mathsf{squash}\ (\mathsf{forward}\ r\ u_0)\ (\mathsf{forward}\ r\ u_1)\ s \\ \mathsf{forward}\ r\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) & = & \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{forward}\ r\ u]\ (\mathsf{forward}\ r\ u_0) \end{array}$$

Using this operation, we can define a general composition operation[3]

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : \|A\| \quad \Gamma \vdash u_0 : \|A(i0)\|\,[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ \ \|A\|\ \ [\varphi \mapsto u]\ u_0 : \|A(i1)\|\,[\varphi \mapsto u(i1)]}$$

by $\Gamma \vdash \mathsf{comp}^i\ \ \|A\|\ \ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{forward}\ i\ u]\ (\mathsf{forward}\ 0\ u_0) : \|A(i/1)\|$.

Propositional truncation, semantical presentation

Given $\Gamma \vdash A$ we define $\Gamma \vdash \|A\|$. For this, we define first an "upper approximation" $\Gamma \vdash X$. An element of $X(I,\rho)$ is of the form $\mathsf{inc}\ a$ with a in $A(I,\rho)$ or $\mathsf{squash}\ u_0\ u_1\ r$ with $r \neq 0, 1$ in $\mathbb{I}(I)$ and u_0 in $X(I,\rho)$ and u_1 in $X(I,\rho)$ or of the form $\mathsf{hcomp}\ [\psi \mapsto u]\ u_0$ with $\psi \neq 1$ in $\mathbb{F}(I)$ and u_0 in $X(I,\rho)$ and u a family of elements u_f in $X(J^+, \rho f p)$ for $f : J \to I$ such that $\psi f = 1$. Each element in $X(I,\rho)$ can be seen as a well-founded tree.

We can then define uf in $X(J, \rho f)$ for u in $X(I,\rho)$ and $f : J \to I$ by induction on u in such a way that $(uf)g = u(fg)$ and $u1_I = u$.

We define then $\|A\|$ to be the subpresheaf of X by taking $\|A\|\,(I,\rho)$ to be the subset of elements $\mathsf{inc}\ a$ or $\mathsf{squash}\ u_0\ u_1\ r$ with u_0 and u_1 in $\|A\|\,(I,\rho)$ and $\mathsf{hcomp}\ [\psi \mapsto u]\ u_0$ with u_0 in $\|A\|\,(I,\rho)$ and $u_f 0 = u_0 f$ and each u_f in $\|A\|\,(J^+, \rho f p)$ and $u_f g^+ = u_{fg}$ for $g : J \to K$.

It is then possible to define a composition structure for $\Gamma \vdash \|A\|$ if we have a composition structure for $\Gamma \vdash A$ exactly as it is done syntactically.

[2]This restriction on the constructor is essential for the justification of the elimination rule, as explained in the Comments at the end.

[3]The open box is given by $\varphi \mapsto u$ and u_0 and it is flattened in the $\|A(i/1)\|$ type by the $\mathsf{forward}$ operation.

Universes

To any Grothendieck universe $\mathcal{U}$, we can associate a corresponding universe U by taking $U(I)$ to be the set of all $\mathcal{U}$-small dependent types $I \vdash A$ with a composition structure. This defines an univalent universe.

Having defined an operation $I \vdash \|A\|$ for $I \vdash A$, we can use the same operation to define a function $U \to U,\ A \mapsto \|A\|$, since $I \vdash \|A\|$ is $\mathcal{U}$-small if $I \vdash A$ is. This means that we have defined an univalent universe which is stable by proposition truncation.

We expect that the same method of defining a composition by "flattening an open box" can be used to define other higher inductive types (suspension, push-out, …). It avoids coherence issues, and an application is that the addition of higher inductive types and univalence to type theory does not raise its proof-theoretic power. Indeed, all we do can be modelled in Aczel's system $\mathrm{CZFu}_{<\omega}$, which is interpretable in type theory with universes.

Comments

Flattening open boxes

One key step is the restriction of the constructor to the form

$$\frac{\Gamma \vdash T \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \|T\| \qquad \Gamma \vdash u_0 : \|T\|\,[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \|T\|\,[\varphi \mapsto u(i1)]}$$

instead of representing directly composition as a constructor (which is what we tried first to implement)

$$\frac{\Gamma, i : \mathbb{I} \vdash T \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \|T\| \qquad \Gamma \vdash u_0 : \|T(i/0)\|\,[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \|T(i/1)\|\,[\varphi \mapsto u(i1)]}$$

Indeed, with this later choice, it does not seem possible to define even a non dependent function $g : \|A\| \to B$ given $f : A \to B$ and $q : \Pi(x\ y : B)B$. We can define $g\ (\mathsf{inc}\ a) = f\ a$ and $g\ (\mathsf{squash}\ u_0\ u_1\ r) = q\ (g\ u_0)\ (g\ u_1)\ r$ but it is not clear how to define $g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)$ since we only know at this point that we have some path $i : \mathbb{I} \vdash T$ such that $A = T(i/1)$ and $u_0 : T(i/0)$ and there is no way to apply an induction for defining $g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)$.

Inductive definition

We have used a generalized inductive definition in the definition of $S^1(I)$. Actually, it is possible to see each element of $S^1(I)$ as a finite object, since a partial element u of extent ψ, which is a family u_f in $S^1(J)$ for each $f : J \to I$ such that $\psi f = 1$, is actually completely determined by the finite set of elements u_f where f is a face map (J is a subset of I and $f(i)$ can only take the value i or 0 or 1).

Suspension

Note that suspension is actually "simpler" than propositional truncation. We define $\mathsf{susp}\ A$ by the rules:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{susp}\ A} \qquad \frac{}{\Gamma \vdash \mathsf{north} : \mathsf{susp}\ A} \qquad \frac{}{\Gamma \vdash \mathsf{south} : \mathsf{susp}\ A} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{merid}\ a\ r : \mathsf{susp}\ A}$$

with the equalities $\mathsf{merid}\ u\ 0 = \mathsf{north}$ and $\mathsf{merid}\ u\ 1 = \mathsf{south}$.

As before, we add composition as a constructor, but only in the form

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{susp}\ A \qquad \Gamma \vdash u_0 : \mathsf{susp}\ A[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{susp}\ A[\varphi \mapsto u(i1)]}$$

Given $x : \mathsf{susp}\ A \vdash B$ and y_N in $B(\mathsf{north})$ and y_S in $B(\mathsf{south})$ and $q : \Pi(x : A)\mathsf{Path}^i\ B(\mathsf{merid}\ x\ i)\ y_N\ y_S$, we define $g : \Pi(x : \mathsf{susp}\ A)B$ by the equations

$$\begin{array}{lcl} g\ \mathsf{north} & = & y_N \\ g\ \mathsf{south} & = & y_S \\ g\ (\mathsf{merid}\ a\ r) & = & q\ a\ r \\ g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) & = & \mathsf{comp}^i\ B(v)\ [\varphi \mapsto g\ u]\ (g\ u_0) \end{array}$$

where $v = \mathsf{hcomp}^j\ [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\ u_0$.

For defining the general composition operation, we define first

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash u : \mathsf{susp}\ A(i/r)}{\Gamma \vdash \mathsf{forward}\ r\ u : \mathsf{susp}\ A(i/1)[(r = 1) \mapsto u]}$$

by the equations

$$\begin{array}{lcl} \mathsf{forward}\ r\ \mathsf{north} & = & \mathsf{north} \\ \mathsf{forward}\ r\ \mathsf{south} & = & \mathsf{south} \\ \mathsf{forward}\ r\ (\mathsf{merid}\ a\ s) & = & \mathsf{merid}\ (\mathsf{comp}^i\ A(i \vee r)\ [(r = 1) \mapsto a]\ a)\ s \\ \mathsf{forward}\ r\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) & = & \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{forward}\ r\ u]\ (\mathsf{forward}\ r\ u_0) \end{array}$$

Using this operation, we can define a general composition operation[4]

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{susp}\ A \quad \Gamma \vdash u_0 : \mathsf{susp}\ A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ (\mathsf{susp}\ A)\ [\varphi \mapsto u]\ u_0 : \mathsf{susp}\ A(i1)[\varphi \mapsto u(i1)]}$$

by $\Gamma \vdash \mathsf{comp}^i\ (\mathsf{susp}\ A)\ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{forward}\ i\ u]\ (\mathsf{forward}\ 0\ u_0) : \mathsf{susp}\ A(i/1)$.

[4]The open box is given by $\varphi \mapsto u$ and u_0 and it is flattened in the $\mathsf{susp}\ A(i/1)$ type by the $\mathsf{forward}$ operation.