HOW TO MEASURE BOREL SETS

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Abstract. The goal of this note is to describe Borel's definition of measure [2]. This definition is not faithfully described in most of the historical account of measure theory. With this definition the *unicity* of measure is no problem, while the *existence* can be expressed as a *coherence* problem. This was clearly recognised by Lusin [6], who formulated this problem as "Borel measure's problem". Lebesgue's definition of measure solves indirectly this problem, but it may be interesting, as suggested by Lusin, to search for a direct solution. We give an example of such a solution.

1. Borel's measure function

This definition appears in "Leçons sur la Théorie des Fonctions", 1898. It is an early example of a *generalised inductive* definition and of a *generalised recursive* definition. We consider only subsets of $(0,1)$. The starting point is the measure of *open* subsets. It was known then that any open can be written as a *countable* union of *disjoint* open intervals (connected components). It is clear that the measure $\mu(r,s)$ of an open interval should be $s-r$. We take then in a natural way the measure of an open set to be the sum of the measure of all its connected components. It was the first satisfactory definition of measure of arbitrary open subsets. Starting from this idea, Borel defines *first* when a subset is measurable (called *well-defined*) and *second* what is its measure. The definition is as follows.

(1) (r,s) is well-defined and $\mu(r,s)$ is $s-r$
(2) If A_n *disjoint* family of well-defined sets $A=\bigcup A_n$ is well-defined, and $\mu A=\Sigma\mu A_n$.
(3) If $A\subseteq B$ are well-defined, $B-A$ is well-defined, and $\mu(B-A)=\mu B-\mu A$.

For instance, the measure of a singleton is 0. Indeed, if $x\in(0,1)$ we have $\mu(0,x)=x$ and $\mu(x,1)=1-x$ and hence $\mu(0,x)\cup(x,1)=x+1-x=1$. Since $\{x\}$ is $(0,1)-((0,x)\cup(x,1)$ it follows that $\mu\{x\}=1-1=0$. It follows that the measure of any countable subset is also 0. We have

$$\mu(0,1/2]=1/2,\ \mu(1/2,3/4]=1/4,$$

$$\mu(3/4,7/8]=1/8,\ \mu(7/8,15/16]=1/16,\ldots$$

and hence

$$\mu((0,1/2]\cup(1/2,3/4]\cup(3/4,7/8]\cup\ldots))=1$$

This definition contains as a special case the measure of an open set

$$\mu((0,1/2)\cup(1/2,3/4)\cup(3/4,7/8)\cup\ldots))=1$$

Notice the *crucial* difference with the usual definition of Borel subsets: the union has to be *disjoint*. In this way, we get a clearly motivated definition With the usual definition (any countable union), this *clear motivation* is *lost*.

Of course, it is then not difficult to show the equivalence with the usual definition (with arbitrary countable union) by a simple induction. The usual definition may read as follow

(1) (r, s) is a Borel set
(2) If A_n family of Borel sets $A = \bigcup A_n$ is a Borel set
(3) If A is a Borel set so is $(0, 1) - A$

It is clear that any well-defined set is Borel. Conversely, it is possible to show by induction that if A and B are well-defined then so is $A \cap B$. If follows that an arbitrary union $A_1 \cup A_2 \cup A_3 \ldots$ can be written as a disjoint union $A_1 \cup (A_2 - (A_1 \cap A_2)) \cup \ldots$ of well-defined sets.

We see then that it is misleading to say the Borel did not prove the *unicity* of the measure, though it is stated in some account of early measure theory that a problem with Borel's definition is that he did not prove unicity nor existence of his notion of measure. With Borel's approach the unicity is *direct*: the clauses (1), (2) and (3) in the definition of well-defined sets specifies in a unique way the measure function.

This is closely connected to the fact that usual presentation of Borel's definition does not stress the point that Borel was using only disjoint unions in his definition. If we start from the second definition of Borel sets, it is indeed not at all clear how to define the measure and why it may be unique. This makes the discovery of Borel less clear and less beautiful than it was.

It is interesting to compare with Jordan-Peano's definition of measure. This definition started first with the measure of finite union of intervals, and then defined the outer and inner measure, but with finite union: the outer measure $\mu^* A$ is the g.l.b. of all measure of finite union of intervals that contain A. The inner measure is then defined as

$$\mu_* A = 1 - \mu^*((0, 1) - A)$$

and a set A is measurable iff $\mu_* A = \mu^* A$. A problem with this definition is that the set R of rationals in $(0, 1)$ is not measurable: indeed we have $\mu^* R = 1$ but $\mu_* R = 0$, because the outer measure of a dense subset has to be 1 and both R and its complement are dense. This problem is solved in a satisfactory and elegant way by Borel. As Bourbaki said, Borel's definition "opens a new era in Analysis".

2. Coherence problem

There is however an important problem with Borel's approach. Lusin [6] listed three problems

(1) Does the sum $\Sigma \mu A_n$ converge in the clause (2) of Borel's definition
(2) Can $\mu B - \mu A$ be negative in the clause (3) of Borel's definition
(3) Is the definition coherent

To give a simple example of the coherence problem we have

$$(0, 1/2] \cup (1/2, 3/4] \cup (3/4, 7/8] \cup \cdots = (0, 1)$$

and there are a priori two ways of computing the measure of the set $(0,1)$. As an open interval it has the measure $1-0=1$. As a disjoint union, we expect also it to have the measure

$$1/2-0+3/4-1/2+\dots = 1/2+1/4+\dots$$

Fortunately, in this case, these two ways are compatible and give the same answer 1. But are we sure that this will always be the case? This is a typical *coherence* problem.

This problem was recognized clearly by Borel. Actually it is part of the general philosophy behind this definition, which originates from Drach. We write "axiomatically" the essential properties that the measure should have This defines a theory of a new object In order to justify the introduction of this new object, it has to be shown that this theory is not inconsistent. Borel cites Drach's exposition of Galois theory as a motivation of such an approach and the importance of the point of view is stressed by Borel . This is remarkably similar (but in 1898!) to Hilbert's notion of ideal elements in proof theory. We shall later on analyse what logical principle is needed to ensure the consistency of this definition.

Did Borel solve this coherence problem? Not quite. He limits himself to a proof of Heine-Borel covering theorem and said later that a complete proof of coherence would have been "long and tedious".

3. Lebesgue solution

The coherence problem was solved indirectly by Lebesgue 1902. The solution is similar to Jordan-Cantor's definition, but uses in a crucial way the correct definition of measure of open set that we have seen above. The outer measure is now the g.l.b. of open supersets,

$$\mu^* A = \bigwedge_{U \ open,\ A\subseteq U} \mu(U)$$

while the inner measure can be defined as

$$\mu_* A = 1 - \mu^*((0,1)-A)$$

Lebesgue says then that A is measurable iff $\mu^*(A)=\mu_*(A)$ and then the measure of A is the common value

$$\mu A = \mu^* A = \mu_* A$$

In this approach, by definition, if A is measurable

$$\mu A = \bigwedge_{U \ open,\ A\subseteq U} \mu(U)$$

Such a measure is called *regular*. All current approach to measure theory, starting from Young (1911), Daniell, Stone, Bourbaki are based on this fundamental idea. One replaces open subsets by *lower semi-continuous* functions, but the essential idea stays the same. The extension theorem attributed to Caratheodory [4] is also based on the idea of using outer measure with countable union of basic sets.

It is quite interesting thus that it has been observed by J.D.M. Wright [**?**] that in some cases of vector-valued measure, the measure *is not* regular. We shall give such an example below over Cantor space. The measure still has the weaker property that a measure of an open is the supremum of the measure of its compact subsets, but there is a subset which is measurable of measure 0 and dense, and so of outer measure 0. This indicates a weakness of the outer measure approach, which cannot thus be used in the case of vector-valued measure. The inductive definition of measure that we present below, following Borel, does not have this problem.

Lebesgue showed that these notions have the required properties of the axiomatic definition of Borel. In particular, this solves the coherence problem of Borel. Furthermore, it can be shown that if A is measurable then one can find well-defined subsets B_1 and B_2 such that

$$B_1 \subseteq A \subseteq B_2$$

and then $\mu A = \mu B_1 = \mu B_2$. Lebesgue changed then the "measurable" of Borel to "B measurable" and Borel changed later on the B-measurable to "well-defined". This stresses the fact that, according to the intuition of Borel, the collection of all Lebesgue measurable sets is a little vague. This intuition was confirmed by work on set theory: it is independent of the usual axiom of set theory whether or not all projective sets (a class of subsets of $(0,1)$ that may seem quite reasonable) are Lebesgue measurable or not.

4. Borel's measure problem

Lusin [6] noticed that there is a difference between Borel's purely inductive definition, and Lebesgue's solution. Cannot we have a direct inductive justification of an inductive definition of measure of Borel sets?? This is Borel's measure problem

We present a solution which is inductive and use only constructive logic.

5. A possible inductive solution

First we reformulate slightly the problem. Instead of working with $(0,1)$ we shall work with Cantor space Ω, the space of all infinite sequence of 0 and 1, space which is important in probability theory. The basic open sets (closed and open subsets) play the role of open intervals. They are finite disjoint union of simple subsets of the form U_σ which is the set of all sequences extending a given finite sequence σ. For instance U_{00} is the set of all sequences starting by 00. We take the measure of U_σ to be 2^{-n} where n is the length of σ and this defines uniquely the measure of all basic open sets. Furthermore the measure of Ω is 1.

It may be interesting to note that this space Ω can be described in purely syntactical term. The collection of basic open sets form a Boolean algebra B that can be described purely syntactically without references to infinite sequences. The measure μ is then a function $B \to [0,1]$ satisfying the fundamental equality, which expresses that μ is a *valuation*

$$\mu(A \cap B) + \mu(A \cup B) = \mu A + \mu B$$

The Boolean algebra B is the Lindenbaum-Tarski algebra of propositional logic.

We can now define in a formal/syntactical way Borel subsets of Ω: it is a symbolic infinitary expression built from simple sets by repeated formal union and intersection

Inclusion can be defined via an infinitary sequent calculus following Novikov,Lorenzen,Schutte. What we get is the Lindenbaum-Tarski algebra of propositional ω-logic (Scott-Tarski). This is the approach taken in Martin-Löf "Notes on Constructive Mathematics" for defining Borel sets.

To take an example, we consider the set of *normal* sequences, which is a Borel subset of Ω. Define $r_i(\omega) = 2\omega_i - 1$ and $s_n = \Sigma_{i\leq n} r_i$ and then

$$b_{n,k} = \{\omega \in \Omega \mid \quad |\frac{s_n(\omega)}{n}| \leq \frac{1}{k} \quad \}$$

which is a simple set $b_{n,k} \in B$ The Borel subset

$$N = \bigwedge_k \bigvee_m \bigwedge_{n\geq m} b_{n,k}$$

is the set of *normal sequences*. We see that it is defined, not as a set of sequences, but as a infinitary symbolic expression. This approach fits with the terminlogy of "well-defined" set, used by Borel.

If k_n is strictly increasing sequence of integers, then

$$N' \subseteq \bigvee_{n\geq m} b'_{n,k_n} \qquad \bigwedge_{n\geq m} b_{n,k_n} \subseteq N$$

An *essential property* of the collection of Borel sets is the following initiality property. This property is known in logic as Rasiowa-Sikorski lemma, or completness of propositional ω-logic. Let B_1 be the σ-algebra of Borel subsets of Ω

Theorem: B_1 is the *free* σ-algebra on B

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & A \\ B_1 \arrow[ur, "\exists ! \bar{f}"'] & \end{tikzcd}$$

We can *define* define the algebra of Borel sets as the free σ-algebra on B. This definition has the following suggestive interpretation: we introduce infinitary symbolic expressions and use *freely* the law of σ-complete Boolean algebras.

We have to show that this does not introduce inconsistency. In "Notes on Constructive Mathematics" this is justified via a cut-elimination theorem, similar to Gentzen's cut-elimination theorem.

This expresses well Borel's intuition. Futhermore it points out towards a way to solve the coherence problem: we should try to define the measure of Borel sets following the *initiality property*. This would solve the coherence problem in an elegant way. Before showing how to do this, we shall express the initiality property in another way, looking at the collection of *bounded Baire functions* over Cantor space Ω instead of the collection of Borel subsets. These subsets can be recovered as the bounded Baire functions taking only values 0 or 1.

6. Measures on Boolean algebras

Already Tarski (1929) showed that it is convenient to "linearize" the problem of measure

We replace the Boolean algebra of basic event by the space of basic random variables $V(B)$

The elements of $V(B)$ can be seen as finite formal sums $\Sigma q_i b_i$

$B \to V(B)$ is the universal valuation!

The measure μ on B can be seen as a positive linear functional $E : V(B) \to Q$ (expectation)

Riesz space

$V(B)$ is an example of a Riesz space

$C(X)$ is another example

Ordered vector space

Any two elements have a sup

One can consider also commutative ordered monoid that are lattices

7. Riesz space

Very basic structure, due to Frederik Riesz (1928)

Rich properties: for instance, any Riesz space is a distributive lattice

Cover very different class of examples: monoid of natural numbers for multiplication and divibility as ordering, and $C(X)$

The basic property

$$x \vee y + x \wedge y = x + y$$

naturally connects with the definition of measure on Boolean algebras

$$\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$$

On a monoid, we define $x \perp y$ iff $x \wedge y = 0$

Euclides' lemma: if $x \leq y + z$ and $x \perp z$ then $x \leq y$

This holds for numbers and for continuous functions!

8. Bounded Baire functions

Strong unit: element 1 such that for any x

$$-n \cdot 1 \leq x \leq n \cdot 1$$

for some n

Dedekind σ-complete: any bounded increasing sequence has a sup

Theorem: the space $B(\Omega)$ of bounded Baire functions on Ω is the σ-completion of $V(B)$.

Baire functions: first continuous functions, then we close by (bounded) pointwise limits

The theorem is quite close to Rasiowa-Sikorski lemma; also very close to completness of propositional ω-logic, and close to Loomis-Sikorski representation of σ-complete algebras

9. How to define measure inductively

We let M_I be the space of functionals l on $V(B)$

$$-nI(f) \leq l(f) \leq nI(f)$$

for $f \geq 0$

We define $I_f \in M_I$

$$I_f(g) = I(fg)$$

Main remark: $I_{f_1 \vee f_2}$ is $I_{f_1} \vee I_{f_2}$

By initiality $f \longmapsto I_f$ extends to $B(\Omega)$

So if f Baire functions and $g \in B(V)$ we can consider $I_f(g)$

In particular $I_f(1)$ is the integral of f

Notice that the initiality states exactly the monotone convergence theorem!

10. Constructive Probability Theory

$$b_{n,k} = \{\omega \in \Omega \mid \quad |\frac{s_n(\omega)}{n}| \leq \frac{1}{k} \quad \}$$

$$N = \bigwedge_k \bigvee_m \bigwedge_{n \geq m} b_{n,k}$$

If k_n strictly increasing

$$N' \subseteq \bigvee_{n \geq m} b'_{n,k_n}$$

Lemma: We can find k_n such that $\Sigma \mu(b'_{n,k_n})$ converges

Theorem: (Borel) $\mu_1(N) = 1$

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