# Geometric Hahn-Banach theorem

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In [MP2] is proved in a constructive way the following result: a point $x$ lies in a compact convex set $K$ in a normed space if and only if it lies within any bound for $K$ (a bound for $K$ being intuitively a parallel pair of hyperplanes between which $K$ lies). This is given as an application of the localic form of the Hahn-Banach theorem, proved in a non constructive way in [MP1]. In [CC] we give a direct constructive proof of Hahn-Banach theorem and using this, we can give a much simpler proof of the characterisation of compact convex sets.

This note is organised as follows. First we recall the direct description the weak* topology on the unit ball of the dual of a normed space given in [CC]. We then use this to characterise the convex closed hull of a totally bounded subset.

## 1 Description of the unit ball in the dual space

Let $E$ be a normed vector space (over the rational numbers). We write $x \in N(q)$ to express that the norm of $x$ is $< q$. We describe a *complete Heyting algebra* by generators and relations. This complete Heyting algebra should be thought of as a point-free description of of the unit ball of the dual $E'$ for the weak* topology. The generators are formal expressions $\lambda x < q$, with $x \in E$, and $q \in \mathbb{Q}$ and the relations are

(1) $[\lambda x < q] \wedge [q < \lambda x] = 0$
(2) $[\lambda(x + y) < r + s] \leq [\lambda x < r] \vee [\lambda y < s]$
(3) $1 = [\lambda x < 1]$ if $x \in N(1)$

where $q < \lambda x$ is defined to be $\lambda(-x) < -q$, together with the continuity axiom

(4) $[\lambda x < q] = \vee_{q'<q}[\lambda x < q']$

Here are simple remarks about $Fn(E)$:

**Proposition 1.1** *It follows from (1) and (2) that we have*

$$[\lambda x < p] \wedge [\lambda y < q] \ \leq \ [\lambda(x + y) < p + q]$$

*If* $0 < r$ *we have* $[\lambda(rx) < rp] = [\lambda x < p]$. *If* $r < s$ *we have* $1 = [\lambda x < s] \vee [r < \lambda x]$. *If* $x \in N(q)$ *we have* $1 = [\lambda x < q]$. *For any* $\epsilon > 0$ *we have*

$$1 = \vee_q \ \ [q < \lambda x] \wedge [\lambda x < q + \epsilon]$$

Notice that we recover the generators $\lambda x \in (p, q)$ used in [MP1] by defining

$$\lambda x \in (p, q) \qquad =_{def} \qquad [p < \lambda x] \wedge [\lambda x < q]$$

but the use of generators $\lambda x < q$ is the key to get a simple descrition of the frame $Fn(E)$ [CC]. In particular, [CC] provides a direct proof that this frame is compact regular, which is the localic form of Alaoglu's theorem, and of the following result[1].

**Theorem 1.2** *In* $Fn(E)$ *we have*

$$1 = [\lambda y_0 < q] \vee \ldots \vee [\lambda y_{m-1} < q]$$

*if, and only if, there exists non negative rationals* $s_j$ *such that* $\Sigma s_j = 1$ *and* $\Sigma s_j y_j \in N(q)$*.*

## 2 Convex hull

We suppose given a totally bounded subset $K$ of $E$. We want to characterise the compact convex hull of $K$.

**Theorem 2.1** *A point* $x$ *belongs to the closed convex hull of* $K$ *if, and only if, we have in the theory* $Fn(E)$ *for all* $q \in \mathbb{Q}$

$$[\lambda x < q] \leq \vee_{y \in K} [\lambda y < q]$$

One possible intuitive reading of this result is that a point belongs to the closed convex hull of $K$ if, and only if, it belongs to all closed regions bounded by hyperplanes that contain $K$[2].

*Proof.* Assume

$$[\lambda x < q] \leq \vee_{y \in K} [\lambda y < q]$$

in $Fn(E)$. Fix $\epsilon > 0$. Since $K$ is totally bounded, there is a *finite* family $y_0, \ldots, y_{m-1}$ of points in $K$ such that for all $y \in K$ we have $y - y_j \in N(\epsilon)$ for some $j < m$. This implies

$$[\lambda x < q] \leq \vee_j [\lambda y_j < q + \epsilon] \qquad (*)$$

for all $q \in \mathbb{Q}$. In the theory $Fn(E)$ we have for each $j$, by Proposition 1.1

$$1 = [\lambda(x - y_j) < 2\epsilon] \vee [\epsilon < \lambda(x - y_j)]$$

But $(*)$ implies, using Proposition 1.1

$$\wedge_j [\epsilon < \lambda(x - y_j)] = 0$$

and hence

$$1 = \vee_j [\lambda(x - y_j) < 2\epsilon]$$

By theorem 1.2 we can conclude that we have non negative $s_j$ such that $\Sigma s_j = 1$ and $\Sigma s_j (x - y_j) \in N(2\epsilon)$.

It follows that $x$ belongs to the compact convex hull of $K$, as desired. □

[1] This is obtained by analysing the distributive lattice which is generated by the relations (1),(2),(3). This distributive lattice is *normal* [CaC], and the space $Fn(E)$ can be defined as the space of maximal points of the spectrum of this distributive lattice.

[2] The statement we give is a little different from the one in [MP2] but seems to correspond in a closer way to the usual statement.

## References

[CC] J. Cederquist and Th. Coquand. Entailment Relations and Distributive Lattices. *Proceeding of Logic Colloquium 1998.*

[CaC] M. Carral and M. Coste. Normal spectral spaces and their dimensions. J. Pure Appl. Algebra 30 (1983), no. 3, 227–235.

[Joh] P. T. Johnstone. *Stone Spaces.* Cambridge studies in advanced mathematics 3, 1982.

[MP1] C.J. Mulvey and J.W. Pelletier. A globalization of the Hahn-Banach theorem. Adv. Math. 89 (1991), no. 1, 1–59.

[MP2] C.J. Mulvey and J.W. Pelletier. A geometric characterization concerning compact, convex sets. Math. Proc. Cambridge Philos. Soc. 109 (1991), no. 2, 351–361.