A Direct Proof of the Localic Hahn-Banach Theorem

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Abstract

Using the notion of *entailment relation* [Sco74] we give a direct proof of the localic version of Hahn-Banach's theorem.

1 Introduction

There are several examples of topological spaces naturally associated to a mathematical structure: the spectrum (with Zariski topology) associated to a ring, the space of valuations associated to a field, and the space $\mathsf{Fn}\ E$ of linear functionals of norm not exceding 1 associated to a normed vector space E. In most of these examples, the associated space is compact (and sometimes compact Haussdorff), and one has an extension theorem of the form: if A is a substructure of B then any point of the space associated to A extend to a point of the space associated to B. It is very natural to analyse such situation from a "topology without points" or localic [Joh82] framework: instead of associating a space, one associates a distributive lattice. This has been done in the case of the spectrum of a ring, by a construction due to A. Joyal [Esp86]. It is shown in [Esp86] how some theorems of commutative algebra become constructive and elementary if understood in such a "pointfree" way. Another suggestive view is to see this distributive lattice as the (Tarski-)Lindenbaum algebra of a propositional theory. The extension theorem mentionned above becomes then a conservativity theorem between two theories.

In this note we analyse the case of the space $\mathsf{Fn}\ E$. A description can be found in the paper of Mulvey and Pelletier [MP91] or in the work of Vermeulen [Ver86], but we think that our analysis is more perspicuous. In particular, a result of our analysis is a direct and elementary proof of the localic version of the Hahn-Banach theorem. The key ingredient is the notion of *entailment relation* due to D. Scott [Sco74] which can be seen either as a way to generate a distributive lattice, or as an abstract generalisation of Gentzen multi-conclusion sequent calculus. To any seminormed space E we associate such an entailment relation where the atomic propositions are pairs (x, r) where $x \in E$ and $r \in Q$. (Roughly this represents the information $u(x) > r$ for a generic u in the dual of E of norm ≤ 1.) Using this relation, we can characterise directly the topology of the space $\mathsf{Fn}\ E$.

2 Entailment Relation

First we give a version for commutative group. Let A be a commuative group. We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ arbitrary finite subsets of A. We define the following relation $\mathcal{A} \vdash \mathcal{B}$ iff there exists a non empty sum of elements of $\mathcal{A}$ equal to a sum of elements in $\mathcal{B}$.

Theorem 2.1 *The relation $\vdash$ satisfies the following conditions of reflexivity, monotonicity and transivity*

$$\mathcal{A} \vdash \mathcal{B} \; if \; \mathcal{A} \cap \mathcal{B} \neq \emptyset \tag{R}$$

$$\frac{\mathcal{A} \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A}' \vdash \mathcal{B}, \mathcal{B}'} \tag{M}$$

$$\frac{\mathcal{A} \vdash x, \mathcal{C} \quad \mathcal{A}, x \vdash \mathcal{C}}{\mathcal{A} \vdash \mathcal{C}} \tag{T}$$

Theorem 2.2 *The relation $\vdash$ is the least entailment relation such that*

$$x + y \vdash x, y$$

$$x, -x \vdash$$

3 SemiNormed Space

We suppose given a seminormed space E with a seminorm $x \in N(q)$ for $x \in E$ and $q \in \mathsf{Q}$. (We shall use freely the concepts and notations presented in the paper [MP91].) We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ arbitrary finite sets of pairs (x, p) with $x \in E$ and $p \in \mathsf{Q}$. We write $p < x$ instead of (x, p). We define then the following relation

$$\{p_1 < x_1, \ldots, p_n < x_n\} \vdash \{q_1 < y_1, \ldots, q_m < y_m\}$$

which is thought of intuitively as

$$\wedge p_i < u(x_i) \vdash \vee q_j < u(y_j)$$

for a generic $u \in \mathsf{Fn}_f\ E$. This is defined by the existence of $r_i \geq 0, r \geq 0$ and $s_j \geq 0$ and of $z \in N(1)$ such that $r + \Sigma r_i > 0$ and $rz + \Sigma r_i x_i = \Sigma s_j y_j$ and $-r + \Sigma r_i p_i = \Sigma s_j q_j$.

The intuition behind this definition is that the only finitary way to have that $p_i < x_i$ and $-1 < x$ for $x \in N(1)$ implies $\vee q_j < y_j$ is to find $s_j \geq 0$ such that $\Sigma s_j q_j < \Sigma s_j y_j$ follows from $p_i < x_i$ and $-1 < x$ for $x \in N(1)$.

We write $\mathcal{A}, \mathcal{B}$ for $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}, p < x$ for $\mathcal{A} \cup \{p < x\}$.

Theorem 3.1 *The relation $\vdash$ satisfies the following conditions of reflexivity, monotonicity and transivity*

$$\mathcal{A} \vdash \mathcal{B} \; if \; \mathcal{A} \cap \mathcal{B} \neq \emptyset \tag{R}$$

$$\frac{\mathcal{A} \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A}' \vdash \mathcal{B}, \mathcal{B}'} \tag{M}$$

$$\frac{\mathcal{A} \vdash p < x, \mathcal{C} \quad \mathcal{A}, p < x \vdash \mathcal{C}}{\mathcal{A} \vdash \mathcal{C}} \tag{T}$$

Thus the relation $\vdash$ is an entailment relation [Sco74].

Proof. The condition (R) and (M) are clear from the definition. The condition (T) can be seen as follows: we have z_1, z_2 of norm < 1 such that

$$rx + \Sigma r_i x_i + t_1 z_1 = \Sigma s_j y_j$$

$$rp + \Sigma r_i p_i - t_1 = \Sigma s_j q_j$$

and

$$\Sigma a_i x_i + t_2 z_2 = bx + \Sigma b_j y_j$$

$$\Sigma a_i p_i - t_2 = bp + \Sigma b_j q_j.$$

We can assume $r > 0$ and $b > 0$, otherwise the conclusion is direct. It follows that

$$\Sigma c_i x_i + (bt_1 z_1 + rt_2 z_2) = \Sigma d_j y_j$$

$$\Sigma c_i p_i - (bt_1 + rt_2) = \Sigma d_j q_j$$

with $c_i = br_i + ra_i$ and $d_j = bs_j + rb_j$. We have furthermore $\Sigma c_i + bt_1 + rt_2 = b(\Sigma r_i + t_1) + r(\Sigma a_i + t_2) > 0$. $\square$

For instance, we have $x + y > p + q \vdash x > p,\ y > q$ and $x > p \vdash rx > rp$ if $r > 0$.

4 An Axiomatisation of $\vdash$

The following is a finite set of axioms that characterises the entailment relation $\vdash$. We write $x < s$ for $-s < -x$.

- (A1) $r < x, x < r \vdash \emptyset$,
- (A2) $r + s < x + y \vdash r < x, s < y$,
- (A3) $\vdash x < 1$ if $x \in N(1)$.

For instance $r < x, s < y \vdash r + s < x + y$ is derivable from these axioms, since $r < x \vdash r + s < x + y, y < s$ and $s < y, y < s \vdash \emptyset$. Similarly, $q < x \vdash p < x$ if $p < q$ because we have then $\vdash p < x, x < q$ and $q < x, x < q \vdash \emptyset$. Similarly we can prove $\vdash 0 < p$ for any $p > 0$ and hence, by $(A2)$ we deduce that $\vdash x < s, r < x$ if $r < s$. Finally, we have $tx > 0 \vdash x > 0$ if $t > 0$.

Proposition 4.1 *The relation $\vdash$ is the least entailment relation satisfying the 3 axioms above.*

Proof. It is clear that any entailment relation satisfying (A1),(A2) and (A3) contains $\vdash$. Conversely, one checks directly that $\vdash$ satsifies (A1),(A2),(A3). $\square$

5 A Direct Description of $\mathsf{Fn}\ E$

We can use the entailment relation $\vdash$ to give a direct description of the topology of $\mathsf{Fn}\ E$.

First, I will show how to interpret the topology $\mathsf{Fn}_f\ E$ which is defined inductively in [MP91]. The rules there can be seen as connecting some and/or-formulae built on top of the atomic formulae of the form $r < x$, interpreting $x < s$ as $-s < -x$ and $x \in (r, s)$ as $r < x \wedge -s < -x$. The fact that $\vdash$ is an assertion relation ensures that the laws of distributive lattices are interpreted correctly, and we need only to check that all axioms are valid in the interpretation given above. There are 8 rules to consider:

- (F1) $\vdash 0 \in (r, s)$ follows from $\vdash r < 0$ and $\vdash 0 < s$ if $r < 0 < s$,
- (F2) $r < 0 \vdash \perp$ if $0 \leq r$ follows from $r < 0 \vdash \emptyset$,
- (F3) $a \in (r, s) \vdash -a \in (-s, -r)$ follows from the definition of $a \in (r, s)$ as the conjunction of $r < a$ and $-s < -a$,
- (F4) $a \in (r, s) \vdash ta \in (tr, ts)$ if $t > 0$ follows from $r < a \vdash tr < ta$,
- (F5) $a \in (r, s) \wedge a' \in (r', s') \vdash a + a' \in (r + r', s + s')$ follows from $r < a, r' < a' \vdash r + r' < a + a'$,
- (F6) $a \in (r, s) \vdash a \in (r, s') \vee a \in (r', s)$ if $r' < s'$ follows from $\vdash r' < a, -s' < -a$,
- (F7) $\vdash a \in (-1, 1)$ if $a \in N(1)$ follows from $\vdash -1 < a$ if $a \in N(1)$,
- (F8) $a \in (r', s') \vdash a \in (r, s)$ if $r \leq r' < s' \leq s$ follows from $r' < x \vdash r < x$ if $r \leq r'$.

The following theorem for instance follows directly from our interpretation. (This proposition is proved by a classical reasoning in [MP91] and as an application of the localic Hahn-Banach in [CCN98, Ver86].)

Theorem 5.1 *If* $\vdash a \in (-1, 1)$ *then* $a \in N(1)$.

Proof. By our interpretation, we have $\vdash -1 < a$ where $\vdash$ is the entailment relation defined above. Hence there exists $s \geq 0$ and $r > 0$ and x such that $x \in N(1)$ and $rx = sa$ and $-r = -s$. It follows that $a = x$ and so $a \in N(1)$. $\square$

So far, we have interpreted the theory of $\mathsf{Fn}_f E$ in the theory of the entailment relation $\vdash$. A natural question is if this interpretation is "complete", so that $\mathsf{Fn}_f E$ is faithfully represented by $\vdash$. For this, we give an interpretation of the axioms $(A1), (A2), (A3)$ in the theory defined by $(F1), \ldots, (F8)$. This interpretation is an example of *approximable relation* between two distributive lattices. In general, this is a relation $R \subseteq D \times L$ such that $\{y \mid R(x, y)\}$ is a filter and $\{x \mid R(x, y)\}$ is an ideal. We define such a relation by relating the atoms: $p < x$ is related to $x \in (p, q)$ iff $x \in N(q)$. It can be shown in the theory $(F1), \ldots, (F8)$ that $x \in (p, q)$ and $x \in (p, q')$ are equivalent if $x \in N(q)$ and $x \in N(q')$.

Theorem 5.2 *The translations of the axioms* $(A1), (A2), (A3)$ *are provable in the theory* $(F1), \ldots, (F8)$.

The theorem of Hahn-Banach is also a direct application of our interpretation. We recall [MP91, CCN98] that we have to define when a basic open $\wedge p_i < x_i$ is covered by a family of basic open $\bigvee_j \wedge_k q_k^j < y_k^j$. This is the case iff for any choice $p_i' > p_i$ we can find a finite number of indices $j_1, \dots, j_n$ and $q_k'^j > q_k^j$ such that for any choice of $k_1, \dots, k_n$ we have

$$\wedge p_i' < x_i \vdash \vee_l q_{k_l}'^{j_l} < y_{k_l}^{j_l}.$$

Intuitively, what we do is to rewrite $\bigvee_j \wedge_k q_k^j < y_k^j$ as $\bigvee_j \bigvee_{q_k'^j > q_k^j} \wedge_k q_k'^j < y_k^j$ and to use the fact that $\wedge p_i' < x_i$ is well-inside the open $\wedge p_i < x_i$ if $p_i < p_i'$. From this characterisation follows directly the localic version of Hahn-Banach.

6 Connection with the Pointwise Version

Since it can be shown generally that compact regular spaces have enough points (using non effective reasoning), we can use our formal version to give a new derivation of Hahn-Banach's extension theorem.

It may be remarked that

$$p_1 < x_1, \dots, p_n < x_n \vdash q_1 < y_1, \dots, q_m < y_m$$

is *not* equivalent to the fact that

$$if\ u(x_1) > p_1\ and\ \dots\ and\ u(x_n) > p_n\ then\ u(y_1) > q_1\ and\ \dots\ and\ u(y_m) > q_m$$

for all $u \in Fn\ E$. For instance if x is of norm exactly one then we cannot have $u(x) > 1$ if $u \in Fn\ E$. However we do not have $x > 1 \vdash$; we only have $x > p \vdash$ whenever $p > 1$.

Extension

The same technique can be used to give a direct description of the space of valuation ring over a field [CP98], of the space of measures over a distributive lattice. In each cases, we obtain a direct localic version of extension theorems. Thus, we think that what should be associated to a mathematical structure is not a space, or a distributive lattice, but directly an entailment relation.

These results seem to confirm the impression that Hahn-Banach's theorem cannot be used in a computational way. The examples whose usual proofs seem to require its use (Runge's theorem or Muntz' theorem) can be rather directly reformulated, using analytic function theory in value in a Banach space, so that to avoid any use of Hahn-Banach. Here is a typical example: if a function $f : \mathsf{C} \to E$ is analytic and bounded then it is constant. One proof uses Hahn-Banach and Liouville's theorem, noticing that any composition $\lambda \circ f$ is bounded for any $\lambda \in \mathsf{Fn}\ E$. The other direct proof consists in following the usual proof of Liouville's theorem, but for E-valued functions.

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