Plan

Lecture 1: Kripke-Joyal, some examples

Lecture 2: one more example, small Zariski topos

Lecture 3: big Zariski topos, light condensed sets; towards higher topos

Big Zariski topos

We have a completness theorem for *coherent* first-order formulae

A formula is true for R iff it is provable from axioms of local rings

Anders Kock noticed that there are non coherent formulae true for R but not consequence of the theory, e.g. $\neg(x = 0) \rightarrow U(x)$

Can we axiomatise the (higher) Zariski topos?

Some notations

Fix a commutative ring R

If A R-algebra write $Sp(A) = Hom_{\mathsf{RAlg}}(A, R)$

Note that $Sp(A[1/u])$ can be seen a subset of $Sp(A)$

Indeed if $x : Sp(A[1/u])$ to give $x : Hom(A[1/u], R)$ is the same as to give $x : Hom(A, R)$ such that $x(u)$ is invertible

$$\begin{array}{ccc} A & \longrightarrow & A[1/u] \\ & \searrow & \vdots \\ & & R \end{array}$$

Big Zariski topos

We use $A, B, \ldots$ for f.p. R-algebras

Axiom 1 R local ring

Axiom 2 (duality) If A f.p. R-algebra then the canonical map $A \rightarrow R^{Sp(A)}$ is an isomorphism

Axiom 3 (local choice) If A f.p. R-algebra and $p : X \twoheadrightarrow Sp(A)$ surjective then there exists $b_1, \ldots, b_n$ comaximal in A and partial sections $s_i : Sp(A[1/b_i]) \rightarrow X$ of p.

See the page web of Felix Wellen

Synthetic Algebraic Geometry

One can develop basic results and concepts of algebraic geometry from these axioms using the language of *dependent type theory*

E.g. definition of schemes, type of schemes and proof that schemes are closed by sigma types!

Synthetic Algebraic Geometry

Consequences of the Duality principle

$\neg(r_1 = \cdots = r_n = 0) \rightarrow U(r_1) \vee \cdots \vee U(r_n)$

Define $A = R/(r_1, \ldots, r_n)$

By duality, $Sp(A) = \emptyset$ iff $1 = 0$ in A iff $1 = (r_1, \ldots, r_n)$ iff one r_i is invertible (since R is local)

Since $R^n = Sp(R[X_1, \ldots, X_n])$ any map $R^n \rightarrow R^m$ is polynomial

Synthetic Algebraic Geometry

Consequence of local choice

$Z(A) = Z(R)^{Sp(A)}$ Zariski spectrum of A

$Z(R)$ is the type of "compact" open proposition

An element of $Z(R)$ is a proposition of the form $U(r_1) \vee \cdots \vee U(r_n)$ for a sequence $r_1, \ldots, r_n$ of elements of R

An open subset of any type X is a family $X \to Z(R)$

Synthetic Algebraic Geometry

A type is *affine* iff it is $Sp(A)$ for some f.p. R-algebra A

An affine type is a (h)set

A scheme is a type which is a finite union of *open* affine subsets!

X such that we have $V_1 : X \to Z(R), \ldots, V_n : X \to Z(R)$ with V_i affine and $X = V_1 \cup \cdots \cup V_n$

So a scheme is defined as a type satisfying some property

A map of schemes is simply an usual map!

Synthetic Algebraic Geometry

If one looks at this externally, this corresponds to a definition of quasi-compact quasi-separated schemes of finite presentation as functor of points

This approach of defining schemes as functor of points was emphasized by Grothendieck early on

Cf. Max Zeuner *Univalent Foundations of Constructive Algebraic Geometry*

Synthetic Algebraic Geometry

Anders Kock noticed 1974 that we can define $\mathbb{P}^n$ as the *set* quotient of $R^{n+1} - 0 = \{(r_0, \dots, r_n) \mid U(r_0) \vee \dots \vee U(r_n)\}$ by the equivalence relation of proportionality

$\Sigma_{r:R^\times}(y_0, \dots, y_n) = r(x_0, \dots, x_n)$

Type of lines in R^{n+1} a line is written $(x_0 : \dots : x_n)$

(This is somewhat surprising: externally $\mathbb{P}^n$ is the following functor

$\mathbb{P}^n(A)$ is the set of sub A-modules of A^{n+1} of rank 1 that are direct factor)

Synthetic Algebraic Geometry

One can show that $\mathbb{P}^n$ is a scheme

Indeed, we have open covering $V_i(x_0 : \cdots : x_n) = U(x_i)$ and $V_i = R^n$ and $R^n = Sp(R[X_1, \ldots, X_n])$ is affine

Synthetic Algebraic Geometry

One can show that the group of automorphisms of $\mathbb{P}^n$ is $PGL_{n+1}(R)$

This is the group of *bijections* of $\mathbb{P}^n$

A map $\mathbb{P}^n \to \mathbb{P}^m$ is given by homogeneous polynomials of the same degree

Synthetic Algebraic Geometry

If we are in setting with *dependent type* and *univalence*

We can form the type of all schemes Schemes $=$ Schemes$_{\mathcal{U}}$

Schemes$_{\mathcal{U}}$ is a subtype of $\mathcal{U}$

Theorem: *Schemes are closed by dependent sums; affine schemes are closed by dependent sums*

If X : Schemes and $Y : X \rightarrow$ Schemes then $\Sigma_X Y$: Schemes

Note that the type Schemes does not depend on the universe

Synthetic Algebraic Geometry

We can define the type of lines $\text{Lines} = \Sigma_{M:\text{RMod}} \|M = R^1\|$

This is a pointed type and delooping of $R^\times$, so it is $K(R^\times, 1)$

Indeed, $R^1 =_{\text{RMod}} R^1$ is $R^\times$

A line bundle on a type X is a function $X \to \text{Lines}$

Synthetic Algebraic Geometry

Usual Picard group of X is defined to be the group of line bundles on X

$\mathsf{Pic}(X) = \pi_0(\mathsf{Lines}^X)$

The result $\mathsf{Pic}(\mathbb{P}^n) = \mathbb{Z}$ can be refined as $\mathsf{Lines}^{\mathbb{P}^n} = \mathbb{Z} \times \mathsf{Lines}$

(This was noticed by Matthias Hutzler)

Curiously, we also have $\mathsf{Lines}^{\mathsf{Lines}} = \mathbb{Z} \times \mathsf{Lines}$

Brouwer continuity and fan theorem

Site: Boolean algebras isomorphic to the one of propositional logic

Covering: B covered by $B[1/e_1], \ldots, B[1/e_n]$ if $e_1, \ldots, e_n$ non trivial FSOI

In this model, we look at the Boolean algebra C generated freely by countably many element; we have $Sp(C) = Hom(C, 2) = 2^{\mathbb{N}}$

Theorem: (Duality principle) $C \to 2^{Sp(C)}$ *is an isomorphism*

All functions from 2^N to itself are uniformely continuous!

Indeed $Sp(C) \to Sp(C)$ is in bijection with $Hom(C, C)$

Brouwer continuity and fan theorem

Stone spaces: dual of Stone spaces

Example: Cantor space, $\mathbb{N}_\infty$ corresponds to Boolean algebra of finite/cofinite subsets of $\mathbb{N}$

$\mathbb{N}_\infty + \mathbb{N}_\infty \rightarrow \mathbb{N}_\infty$ surjective but no continuous section

Corresponds to an injection between Boolean algebras with no retraction

Light Condensed Sets

The previous topos is similar to the Zariski topos

We now take for the base category all *countably presented* Boolean algebras

Covering: B covered by $B[1/e_1], \ldots, B[1/e_n]$ if $e_1, \ldots, e_n$ FSOI

and we add for covering any *injective* map $B \to B'$

This is the topos of *light condensed sets*, introduced by Clausen and Scholze

We present a possible axiomatisation of this topos

Light Condensed Sets

Axiom 1 (duality): $B \to 2^{Sp(B)}$ is an isomorphism

Axiom 2 (formal surjection): if $B \to B'$ is injective, the corresponding map $Sp(B') \to Sp(B)$ is surjective

Axiom 3 (local choice): if $p : X \twoheadrightarrow Sp(B)$ is surjective there exists $q : Sp(B') \to Sp(B)$ surjective and $s : Sp(B') \to X$ such that $p \circ s = q$

Axiom 4 (dependent choice): given a sequence $X_{n+1} \to X_n$ of surjective maps then all maps $\varprojlim X_n \to X_p$ are surjective

Light Condensed Sets

Some examples of internal/synthetic reasoning

Define a set S to be *Stone* iff there exists a countably presented Boolean algebra B such that S and $Sp(B)$ are in bijection

Any map $S' \to S$ is dual of a map $Hom(B, B')$ so it is uniformely continuous!

As a special case, 2^N and $\mathbb{N}_\infty$ are Stone

Light Condensed Sets

$\mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty,\ inl(k) \mapsto 2k,\ inr(k) \mapsto 2k+1$ is surjective

Hence we have LLPO!

On the other hand we have ¬WLPO since all maps $\mathbb{N}_\infty \to \mathbb{N}_\infty$ are continuous

Logical Principle

WLPO is $\neg(\forall_n b_n = 0) \vee \forall_n b_n = 0$

LPO is $(\exists_n b_n = 1) \vee \forall_n b_n = 0$

LLPO can be formulated as: $(\forall_n b_n = 0) \vee (\forall_m c_m = 0)$ is equivalent to $\forall_{n,m}(b_n = 0 \vee c_m = 0)$

Light Condensed Sets

Axiom 2 can be formulated as $Sp(B)$ inhabited if $\neg(1 = 0)$ in B

This can be seen as completness of (countable) propositional logic

A countably presented Boolean algebra is like a countable propositional theory

Light Condensed Sets

We have Markov's Principle by duality

Let b_n be a binary sequence

Take the Boolean algebra B quotient of 2 with the presentation $b_n = 0$

$B \rightarrow 2^{Sp(B)}$ is a bijection

Hence $1 = 0$ in B (i.e. $1 = b_n$ for some n) iff $Sp(B)$ empty iff $\neg\forall_n b_n = 0$

Light Condensed Sets

Define a proposition to be closed iff it is of the form $\forall_n a_n$ with a_n decidable

Define a proposition to be open iff it is of the form $\exists_n b_n$ with b_n decidable

Theorem: *a countable intersection of decidable subsets of a Stone set S is exactly a family of closed propositions*

Using Markov's Principle, the negation of a closed proposition is open

Light Condensed Sets

Define a set to be Compact Hausdorff iff it is a quotient of a Stone set by a closed equivalence relation

E.g. the unit interval $[0,1]$ is described as a closed quotient of Cantor space

Any such quotient comes with a "topology": family of open propositions

Universes

We define the type Stone of all Stone spaces and the type CHaus of all compact Hausdorff spaces (they are groupoids)

We can show that X : CHaus is Stone iff any connected component of a point is a singleton

Theorem: CHaus *is closed by dependent sums;* Stone *is closed by dependent sums*

If X : CHaus and $Y : X \to$ CHaus then $\Sigma_X Y$: Schemes

Note that the type Schemes does not depend on the universe

Towards Higher Topos

The axioms for the Zariski and light condensed sets made sense in dependent type theory, reading the axiom of local choice (and dependent choice) with *arbitrary* types

In this context, we can use "higher types", i.e. types that are not (h)sets

For instance, we can use Eilenberg-McLane spaces $K(\mathbb{Z},n)$ and propositional truncation to define cohomology, and prove results such as

One defines $H^n(X,\mathbb{Z})$ to be $\pi_0(K(\mathbb{Z},n)^X)$

Theorem: *We have* $H^n([0,1],\mathbb{Z}) = 0$ *if* $n > 0$ *and* $H^1(S_1,\mathbb{Z}) = \mathbb{Z}$ *and* $H^n(S_1,\mathbb{Z}) = 0$ *if* $n > 1$

Towards Higher Topos

Each compact Hausdorff space X has a "resolution" by Stone spaces

$X \leftarrow S \leftleftarrows S \times_X S \ldots$

We can associate to this a cochain complex $C(X, S)$

$\mathbb{Z}^S \rightarrow \mathbb{Z}^{S \times_X S} \rightarrow \mathbb{Z}^{S \times_X S \times_X S} \ldots$

Theorem: $H^n(X, \mathbb{Z})$ *is the* n*th cohomology group of the chain complex* $C(X, S)$

This is a result of Roy Dyckhoff 1976, but we get a purely internal proof

Some analogy

One discovers an analogy

Stone corresponds to the type of affine spaces

CHaus corresponds to the type of separated schemes

E.g. the argument showing closure by sigma types

Topos as generalised set theory

His confidence in the project was strengthened by Dana Scott's work on Boolean valued models, which he heard about at a meeting that same spring at Oberwolfach. Even here it was not the set theoretic aspect of the work that caught Lawvere's attention but the logical aspect. He has said the independence proofs in ZF were less important to him than a paper in which Scott proved the continuum hypothesis independent of a kind of third order theory of the real numbers, because, Scott says: 'once one accepts the idea of Boolean values there is really no need to make the effort of constructing a model for full transfinite set theory' (Scott [1967], p. 109).

To Lawvere this seemed not only simpler than the version for ZF but more to the point.

Type theory and set theory

It is a pity that a system such as Zermelo-Fraenkel set theory is usually presented in a purely formal way, because the conception *behind it is quite straightforwardly based on type theory. One has the concept of an arbitrary* subset of a given domain *and that the collection of* all subsets *of the given domain can form a new domain (of the next type!). Starting with a domain of individuals (possibly empty), this process of forming subsets is then iterated into the* transfinite*. Thus, each set has a type (or* rank*), given by the ordinal number of the stage at which it is first to be found in the iteration.*

Dana Scott, *A type-theoretical alternative to ISWIM, CUCH, OWHY*, 1969

Sheaf models and Universes

Martin-Löf's original motivation for extending simple type theory with a universe

The simple theory of finite types, although proof theoretically quite strong, has some unnatural limitations (for example, it permits only finite iterations of the power operation) and, above all, it is not adequate for a formalization of mathematics that talk about arbitrary sets and not just sets of natural numbers, sets of sets of natural numbers, and so on.

Sheaf models and Universes

it is natural to look at the interpretation of universes in sheaf models

The same question holds for forcing models

Something quite interesting happens then

Generalization of forcing for universes?

How to interpret universes?

Type theory/set theory

Gödel/Tarski formulation of simple type theory: only types $0,\ 1,\ 2, \ldots$, with $n+1$ type of subsets of type n and 0 type of individuals

Set theory: start with 0 empty set and iterate power set transfinitely

See *A Formal Proof of the Independence of the Continuum Hypothesis*, J. M. Han, F. van Doorn, 2021

Generalization of forcing for universes?

This question appeared in algebraic geometry in the 60s, and was a direct motivation of the notion of *stacks*

Cf. *Éléments de Géométrie Algébrique, 1*, 3.3.1, A. Grothendieck and J. Dieudonné

Similar questions appear when one wants to patch together *structures*, as opposed to elements (e.g. value of a function), e.g. in Weil's definition of manifold

Univalence and Sheaf Models

Let A_i be a family of (h)propositions in a type theory with univalence

A type X is a *sheaf* iff each diagonal map $X \to X^{A_i}$ is an equivalence

In presence of univalence the universe of sheaves is itself a sheaf!

Indeed we have $\pi : \mathcal{U}^A \to \mathcal{U},\ B \to \Pi_A B$

We also have $\delta : \mathcal{U} \to \mathcal{U}^A$ diagonal map and $(\pi \circ \delta)X$ is equal to X if $X \to X^A$ is an equivalence *because of univalence*