A Finitary Version of System F

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System F

Introduced by Girard (1970)

Inspired by the presentation of Dialectica interpretation in Shoenfield (1967)

Gödel System T

Types: $N,\ r \to s$

Terms: typed λ-terms + primitive recursion at any types

$$v : N \to s \quad v\ 0 = u \quad v\ (S\ x) = u'\ (v\ x)\ x$$

Reducibility

Very clear proof that any closed term of type N is convertible to a numeral $S^k\ 0$

Reducible at type N: convertible to a numeral

Reducible at type $r \to s$: $v\ u$ is reducible at type s if u reducible at type s

Extensions

Can be extended with a type of ordinals O

$$0 : O, \qquad S : O \to O, \qquad Lim : (N \to O) \to O$$

System T and PA and HA

Ordinals and ID_1

System F

Allow type variables and universal quantification on types

$$\alpha,\ r \to s,\ (\Pi\alpha)r(\alpha)$$

No primitive types

Syntactical representation of the intuitive notion of *uniform/generic* operation

Introduced independently by Reynolds (1974) for this purpose

If $u : (\Pi\alpha)r(\alpha)$ then $u\ s : r(s)$

Numerical functions

One can follow an idea of Church and *define*

$$N = (\Pi\alpha)\alpha \to (\alpha \to \alpha) \to \alpha$$

A natural number is a generic iteration

Exponentiation can be typed

$$mult : N \to N \to N$$
$$n\ N\ (S\ 0)\ (mult\ m) = m^n$$

Reducibility candidate

To be reducible at type $T = (\Pi\alpha)\ \ \alpha \rightarrow \alpha$??

If s is a type and u is reducible at type s then $u\ s\ v$ is reducible at type s

This is *circular*: s can be T itself

Solution: reducibility candidate

Tait's presentation

Λ set of all *untyped lambda-terms* (maybe open)

Equality is β-conversion

Think of a *type* as a *subset* of Λ

If $u : (\Pi\alpha)r(\alpha)$ then $u : r(T)$ for any type T

Π is simply intersection!

Normalisation proof

If $X, Y \subseteq \Lambda$ define

$$X \to Y = \{u \mid v \in X \to u\ v \in Y\}$$

$(\Pi\alpha)\ \ \alpha \to \alpha$ is interpreted by

$$\bigcap_{X \subseteq \Lambda} (X \to X)$$

Π^1_1 comprehension

A priori this definition is also *circular*. We define a *subset* of Λ by quantification over *all* subsets.

In *this* case, the circularity is only *apparent*

One can show that

$$\bigcap_{X \subseteq \Lambda} (X \to X)$$

exists and is

$$\{u \in \Lambda \mid u = \lambda x\ x\}$$

(don't forget that equality is β, η-conversion)

First restriction of system F

No universal quantification

We introduce a primitive type T

If $\Gamma \vdash t : \alpha \to \alpha$ and α not free in Γ then $\Gamma \vdash t : T$

If $\Gamma \vdash t : T$ then $\Gamma \vdash t : s \to s$ for any type s

This system looks circular but it has a finitary interpretation

Π^1_1 comprehension

The same holds with

$$N = (\Pi\alpha)\alpha \to (\alpha \to \alpha) \to \alpha$$

Once can show that

$$\bigcap_{X \subseteq \Lambda} X \to (X \to X) \to X$$

exists and is

$$\{u \mid (\exists n)\ u = \lambda x \lambda y\ y^n\ x\}$$

Notice that here we need quantification over natural numbers

Second restriction of system F

No universal quantification

We introduce a primitive type N

If

$$\Gamma \vdash t : \alpha \to (\alpha \to \alpha) \to \alpha$$

and α not free in Γ then $\Gamma \vdash t : N$

If $\Gamma \vdash t : N$ then

$$\Gamma \vdash t : s \to (s \to s) \to s$$

for any type s

This system looks circular but it has a finitary interpretation

The functions $N \to N$ are exactly the one of Gödel system T!!

How far can we go??

The same idea works for

$$(\Pi\alpha)r(\alpha)$$

if $r(\alpha)$ is of rank ≤ 2

Our main result is that it works for *any*

$$(\Pi\alpha)r(\alpha)$$

where $r(\alpha)$ uses only α and $\rightarrow$

We don't know of any direct proof. We use Kripke models

Proof Theory

Something similar was observed in proof theory

One can *explain* the apparent circular Π^1_1-comprehension by using intuitionistic theory of inductive definitions

In detail, one needs to iterate inductive definitions ID_ω

But the "first level" is explained by ID_1

First proof (Takeuti+Kreisel) rather indirect

Elegant argument by Buchholz (1977)

Fragment of system F

Here we have HA/PA instead of ID_1

To get ID_1 iterate the idea: allow only type

$$(\Pi\alpha)r(\alpha)$$

if $r(\alpha)$ uses only α and $\rightarrow$ and a type of the previous level like T or N

For instance

$$O = (\Pi\alpha)\alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow ((N \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$$

In this way, representation of $ID_1,\ ID_2, \ldots$ in system F

Can we represent ID_ω??