Constructive models of univalence

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Content

(1) Cubical set model, as formalized by Mark Bickford in NuPrl (in particular definition of the universe and proof that it is fibrant)

(2) Constructive Quillen model structure, found by Christian Sattler

(3) Application of this constructive model, new internal model, stack model

The theme of (1) and (2) is: *structure* versus *property*

Cubical sets

On the relation between the fundamental group of a space and the higher homotopy groups
Eilenberg 1939, Fund. Math.

Lemma: *If A subpolyhedra of B then $B \times 0 \cup A \times \mathbb{I}$ is a retract of $B \times \mathbb{I}$*

where $\mathbb{I}$ is the closed unit interval

Proofs of basic results about homotopy “can be obtained quite neatly by repeated, and sometimes tricky, use of this general lemma” (Bourbaki's notes on homotopy, 1951)

Proposition: *Given two homotopic functions f_0 $f_1 : A \to X$ and an extension $f'_0 : B \to X$ of f_0 there is an extension f'_1 of f_1 homotopic to f_0*

Cubical sets

This property was taken as a *starting point* by Kan

1955 *Abstract homotopy I*, cubical sets

1958 *A combinatorial definition of homotopy groups*, simplicial sets

Property of X: if we assume it for B cube and A boundary then the extension homotopy property holds for any B and any A subcubical set of B

Constructive version

One cannot use Kan's characterisation in a constructive setting

If one takes the definition as it is, e.g. in IZF, it is impossible to show

-if $E \rightarrow B$ Kan fibration and b_0 and b_1 are path connected then $E(b_0)$ is weakly equivalent to $E(b_1)$

-If Y has the Kan extension property then so does Y^X (Moore's Theorem)

We would like to have a model of the univalence axiom where we can actually ⟪run⟫/compute with the use of this axiom

Constructive version

We start from a base category

Objects: finite sets $I, J, \ldots$

I represents a formal version of $[0,1]^I$

A *cubical set* is a presheaf on the base category

All this can be defined in NuPrl or in CZF$^+$u$_{<\omega}$

At the end, everything can be formulated in a *nominal* extension of λ-calculus (cf. Simon Huber's work)

Constructive version

A cubical set is an abstract version of a «polyhedra»

We get a concrete version of *subpolyhedra* by considering the cubical set of faces $\mathbb{F}$, a subobject of Ω

$\mathbb{F}(I)$ is the distributive lattice generated by formal symbols $(i = 0), (i = 1)$ and the relation $0 = (i = 0) \wedge (i = 1)$.

The homotopy extension property can then be reformulated as

If $\sigma : A \to B$ is classified by $\mathbb{F}$ any map from $A \times \mathbb{I} \cup B \times 0$ to X can be extended to a map from $B \times \mathbb{I}$ to X

Constructive version

This is a *property* of a cubical set X

We have a quantification over *all* cubical sets B, so a priori this property may not be 《absolute》, i.e. may not be stable by addition of new universes

Theorem: *For a given cubical set* X *there exists* one *particular mono* $A \to B$ *classified by* $\mathbb{F}$, *built effectively from* X, *such that, if* X *satisfies the extension property w.r.t. to this mono, then it satisfies the extension property w.r.t. any mono classified by* $\mathbb{F}$

Furthermore, such an extension corresponds to a *uniform structure* of extensions w.r.t. mono classified by an element of $\mathbb{F}(I)$

Constructive version

The same result holds for the notion of *fibration*

Definition: *A map* $Y \to X$ *is a fibration iff it has the right lifting property w.r.t. any map* $A \times \mathbb{I} \cup B \times 0 \to B \times \mathbb{I}$ *with* $A \to B$ *mono classified by* $\mathbb{F}$

It is enough to have this for *one* particular mono, and then we have a uniform right lifting structure w.r.t. mono classified by an element of $\mathbb{F}(I)$

Constructive version

A map $Y \rightarrow X$ is a *trivial fibration* iff it has the right extension property w.r.t. any mono classified by $\mathbb{F}$

Special case: Y is *contractible* iff any map $A \rightarrow Y$ can be extended to a map $B \rightarrow Y$

It is enough to look at the particular case where $B(I)$ is the set (ψ, u) with ψ in $\mathbb{F}(I)$ and $u : I, \psi \rightarrow Y$ and A is the subobject of B classified by the first projection

Type family

A context $\Gamma, \Delta, \ldots$ is interpreted by a(n arbitrary) cubical set

If Γ is a cubical set, we can consider its *category of elements* $\int \Gamma$

An object is I, ρ with ρ in $\Gamma(I)$

If $f : J \rightarrow I$ we have $f : J, \rho f \rightarrow I, \rho$

Definition : *A type family on* Γ *is a presheaf on* $\int \Gamma$

Notation: $\Gamma \vdash A$

Type family

If $\mathcal{E}$ is the category of cubical sets then $\mathcal{E}/\Gamma$ is *equivalent to* (but not isomorphic to) $\int \Gamma$

Type family

We can define $\Gamma.A$ and the projection $p_A : \Gamma.A \to \Gamma$

A *Kan structure* for A is a fibration structure for p_A

If $\sigma : \Delta \to \Gamma$ we define $\Delta \vdash A\sigma$, and any Kan structure for A can be transported to a Kan structure for $A\sigma$

Remark: *No coherence issues for presheaf models of type theory*

⟪Glueing⟫ operation

Given

-a type family $\Gamma \vdash A$

-a monomorphism $\sigma : \Delta \to \Gamma$ classified by $\mathbb{F}$

-$\Delta \vdash w : T \to A\sigma$

(1) we can define $\Gamma \vdash G$ such that $G\sigma = T$ (*strict* equality)

(2) $\Gamma \vdash e : G \to A$ such that $e\sigma = w$

(3) if T, A furthermore have a Kan structure, and w is an equivalence, then we can find a Kan structure on G which extends the one of T and an equivalence structure on e which extends the one of w

⟪Glueing⟫ operation

This has been formalized by Mark Bickford in NuPrl

Abstract formulation in an arbitrary topos (Ian Orton and Andy Pitts)

Universe

For each Grothendieck universe $\mathcal{U}$, we can associate a universe U

$U(I)$ is the set of $\mathcal{U}$-types $I \vdash A$ *together with* a Kan structure c_A

We can then define $U \vdash El$ by taking $El(I, A, c_A)$ to be the set of all sections $I \vdash u : A$.

Theorem $U \vdash El$ *has a canonical fibration structure* c_E *such that if* $\Gamma \vdash A$ *is a* $\mathcal{U}$*-dependent type* with a fibration structure c_A, *there exists a unique map* $|A| : \Gamma \rightarrow U$ *such that* $El|A| = A$ *and* $c_E|A| = c_A$.

Universe

The universe we consider is essentially *different* from the one of the simplicial set model, where A in $U(I)$ is $I \vdash A$ satisfying the Kan extension *property*

Notice however that if p_A has the right lifting *property* w.r.t. any «open box» mono then $\Gamma \vdash A$ has automatically at least one Kan structure c_A and we can find $|A| : \Gamma \to U$ such that $El|A| = A$ and $c_E|A| = c_A$

Universe

Also formalized by Mark Bickford, who formalized as well

Theorem: U *has a Kan structure*

In cubical type theory we formalized that

Theorem: $A : U \vdash \Sigma(X : U)$ Equiv $A\ X$ *is a trivial fibration*

which is a possible formulation of univalence

Application

Theorem: *If* $\sigma : \Delta \to \Gamma$ *has the left lifting property w.r.t. any fibration and* $\Delta \vdash B$ *is a* $\mathcal{U}$*-dependent type with a fibration structure* c_B *there exists a* $\mathcal{U}$*-dependent type* $\Gamma \vdash A$ *with a fibration structure* c_A *such that* $A\sigma = B$ *and* $c_A\sigma = c_B$.

This is a refinement of the result that fibrations can be extended along trivial cofibrations

A key property of Quillen model structures

Quillen model structure

We have already defined the notion of *fibration* and *trivial fibration*

We define the notion of *cofibration*: this is exactly a map classified by $\mathbb{F}$

We define the notion of *trivial cofibration*: a map which has the left lifting property w.r.t. any fibration

Notice that a priori this notion of trivial cofibration may not be absolute

On the other hand, the notions of fibration, trivial fibration and cofibration are all absolute

Quillen model structure: factorization

Theorem: *Any map* $\sigma : A \to B$ *can be factorized in*

-a trivial cofibration i *followed by a fibration* p

-a cofibration j *followed by a trivial fibration* q

The second factorization is particularly simple

The first factorization is justified by a (finitary) inductive definition

Theorem: (Christian Sattler) *The map* p *is a* trivial *fibration iff the map* j *is a* trivial *cofibration*

Quillen model structure: factorization

First factorization: we define $E(I)$ and $p : E(I) \rightarrow B(I)$ by induction on I. An element of $E(I)$ is of the form

-$i\ a$ with a in $A(I)$ and $p\ (i\ a) = \sigma\ a$ or

-$u_1 = \mathsf{comp}\ b\ (\psi, u)\ u_0$ with b in $E(I^+)$ and we define $p\ u_1 = b\ 1$ with u in $I^+, \psi \rightarrow A$ such that u_0 extends $u\ 0$ and $p\ u = b$ and $p\ u_0 = b\ 0$

Second factorization: an element of $E'(I)$ is of the form (ψ, b, a) with b in $B(I)$ which extends σa with a in $I, \psi \rightarrow A$

Quillen model structure: weak equivalence

Definition: *The map* σ *is a weak equivalence iff* p *is a trivial fibration iff* j *is a trivial cofibration*

Theorem: (Christian Sattler) *A fibration is a weak equivalence iff it is a trivial fibration; a cofibration is a weak equivalence iff it is a trivial cofibration*

It follows from this that the notion of trivial cofibration is absolute

Quillen model structure: weak equivalence

We have already seen that fibrations can be extended along trivial cofibrations

We have defined three notions of map

fibration, cofibration, weak equivalence

This defines (in a constructive meta theory) a Quillen model structure

Applications

Using the factorization cofibration-trivial fibration we can define a new type Id $A\ a\ b$ which satisfies all laws of Martin-Löf identity type (Andrew Swan)

The model can be described in $\mathsf{CZF}^{+}\mathsf{u}_{<\omega}$

Theorem: *The univalence axiom does not add any proof theoretic power to type theory*

Applications

If we use a model of set theory where Markov's principle does not hold

Theorem: *The univalence axiom does not prove Markov's Principle*

If we use a model of set theory where any numerical function on Cantor space is uniformly continuous

Theorem: *The univalence axiom is compatible with continuity principle on Cantor's space*

I expect this to hold for Baire space as well, using propositional truncation for expressing continuity

Applications

What happens with countable choice, formulated with propositional truncation?

Remark: *Countable choice always holds in the groupoid model even if it does not hold in the constructive meta theory*

In order to get a counter model to countable choice, we can use the notion of *stack* (with a suitable formulation; j.w.w. Bassel Mannaa and Fabian Ruch)

Question: how to extend the notion of stack to the cubical set model?

A new internal model of type theory

The work

The next 700 syntactical models of type theory, CPP 2017
S. Boulier, P.-M. Pédrot and N. Tabareau

stresses the importance of syntactical internal models of type theory

《Slice》 model: a type is interpreted by a type in a given context

《Relational》 model: a type is interpreted by two types A and B and a family over $A \times B$

《Predicate》 model: a type is interpreted by a type and a family over this type

Univalence holds in these internal models as soon as it holds in the type theory

A new internal model of type theory

In cubical type theory, we can consider the model where a type is interpreted by two types A and B and a *path* connecting A and B

Theorem: *This forms a model of type theory with univalence and propositional truncation*

We can consider variations of this model, e.g. two types A and B with a path connecting A and B and a family over A and a family over B

This corresponds to the following ∞-stack model: we consider two presheaf models over Sierpinski spaces $U \supseteq W_0$ and $V \supseteq W_1$ and we identify W_0 and W_1