Normalisation for extension types

We treat only the case of the extension type $A[\psi \to a]$ and universe. The treatment of dependent product and sum is the same as before. Note that path types can be defined from extension types: the type of path between a and b can be defined as $\Pi_{i:\mathbf{I}} A[i = 0 \to a,\ i = 1 \to b]$.

For A in $\mathsf{Elem}(U_n)$ we have $\mathsf{Var}(A) \subseteq \mathsf{Neut}(A) \subseteq \mathsf{Norm}(A)$. If e in $\mathsf{Norm}(A)$ we write $\langle e \rangle$ its interpretation in in $\mathsf{Elem}(A)$.

An element of $\mathsf{Neut}(A)$ is x or $k\ u$ with k is $\mathsf{Elem}(\Pi_C D)$ and u in $\mathsf{Norm}(C)$ and $D\langle u \rangle = A$, or $k.1$ with k in $\mathsf{Elem}(\Sigma_C D)$ and $C = A$ or $k.2$ with k in $\mathsf{Elem}(\Sigma_C D)$ and $D(\langle k \rangle .1) = A$.

If u in $\mathsf{Elem}(A)$ then $\mathsf{Norm}(A)|u$ is the set of e in $\mathsf{Norm}(A)$ such that $\langle e \rangle = u$.

If k in $\mathsf{Neut}(A)$ and e in $\mathsf{Norm}(A)|\langle k \rangle$ on ψ then $k|\psi \to e$ in $\mathsf{Norm}(A)$ with $\langle k|\psi \to e \rangle = \langle k \rangle$.

0.1 Extension type

If $T = A[\psi \to u]$ we define $T'(v)$ to be $A'(v)[\psi \to \overline{u}]$

We define $\alpha_T\ v\ \overline{v}$ for $v : T$ and $v = u$ on ψ and $\overline{v} : A'(v)$ and $\overline{v} = \overline{u}$ on ψ as $e = \alpha_A\ v\ \overline{v}$ in $\mathsf{Norm}(A)$. This is an element equal to $\alpha_A\ u\ \overline{u}$ on ψ, and so an element such that $\langle e \rangle = u$ on ψ, and thus it is in $\mathsf{Norm}(T)$.

We now have to define $\beta_T(k, \varphi \to \overline{v})$ which should be in $T'(v)[\varphi \to \overline{v}]$, for k in $\mathsf{Neut}(T)$, which means k in $\mathsf{Neut}(A)$ and $\langle k \rangle = u$ on ψ, and $\overline{v}$ in $T'\langle k \rangle$ on ψ. We take it to be

$$\beta_T(k, \varphi \to \overline{v}) = \beta_A(k, \varphi \to \overline{v}, \psi \to \overline{u})$$

0.2 Universe

If A in $\mathsf{Elem}(U_n)$ we define $U'_n(A)$ as the type of tuples $A', A_0, \alpha_A, \beta_A$ with

1. $A'(u)$ in $\mathcal{U}_n$ for u in $\mathsf{Elem}(A)$
2. A_0 in $\mathsf{Norm}(U_n)|A$
3. $\alpha_A\ u$ in $A'(u) \to \mathsf{Norm}(A)|u$ for u in $\mathsf{Elem}(A)$
4. $\beta_A(k, \psi \to \overline{u})$ in $A'\langle k \rangle[\psi \to \overline{u}]$ if $\overline{u}$ in $A'\langle k \rangle$ on ψ for k in $\mathsf{Neut}(A)$

If $\psi = \perp$ we simply write $\beta_A(k)$ instead of $\beta_A(k, \psi \to \overline{u})$.

We define $\alpha_{U_n}\ A\ (A', A_0, \alpha_A, \beta_A)$ to be A_0.

One main issue seems to be how to define $\beta_{U_n}(K, \psi \to (T', T_0, \alpha_T, \beta_T))$. We can then define the interpretation $\overline{U_n}$ as $U'_n, U_n, \alpha_{U_n}, \beta_{U_n}$.

For this we use in the meta theory the operation of extension of functions of a fixed codomain (also called "Glue" and "unglue"): if A is a type, and $u : T \to A$ a function only defined on ψ we can form the total type $\mathsf{Ext}(u)$ which extends T and $\mathsf{ext}(u) : \mathsf{Ext}(u) \to A$ which extends u. If a in

A and t in T on ψ with $u\ t = a$ on ψ we can form (a, ψ, t) in $\mathsf{Ext}(u)$ such that $(a, \psi, t) = t$ on ψ and $\mathsf{ext}(u)\ (a, \psi, t) = a$.

We take $\beta_{U_n}(K, \psi \to (T', T_0, \alpha_T, \beta_T))$ to be $(X', X_0, \alpha_X, \beta_X)$ with the following definitions.

-$X'(u)$ is $\mathsf{Ext}(\alpha_T\ u)$, so that we have $X'(u) = T'(u)$ on ψ.

-X_0 is $K|\psi \to T_0$, so that we have $X_0 = T_0$ on ψ.

-$\alpha_X\ u$ is $\mathsf{ext}(\alpha_T\ u)$. Note that $\alpha_T\ u$ is of type $T'(u) \to \mathsf{Norm}(K)|u$ and then $\mathsf{ext}(\alpha_T\ u)$ is of type $X'(u) \to \mathsf{Norm}\langle K\rangle|u$.

-$\beta_X(k, \varphi \to \overline{u})$ is $(k_1, \varphi, \overline{u_1})$ with the following definitions. We take $\overline{u_1}$ to be $\beta_T(k, \varphi \to \overline{u})$ in $T'\langle k\rangle[\varphi \to \overline{u}]$ on ψ. We take k_1 to be $k|\varphi \to \mathsf{ext}(\alpha_T\ \langle k\rangle)\ \overline{u}, \psi \to \alpha_T\ \langle k\rangle\ \overline{u_1}$.