AXIOMATIC SET THEORY 3

Reflection Principle

1. Let M_α be an increasing sequence of sets such that each M_α is countable for $\alpha < \omega_1$, and $M_\lambda = \cup_{\alpha<\lambda} M_\alpha$ for λ limit. We write $M = M_{\omega_1}$. Show that for any formula ϕ there exists $\alpha < \omega_1$ such that

$$\phi^{(M)} \leftrightarrow \phi^{(M_\alpha)}$$

2. We work in ZF minus the axiom of infinity and replacement but we add the following form of reflection principle schema

$$(\exists V)[\phi \rightarrow \phi^{(V)}]$$

(here ϕ is a formula that may have free variables; notice that this is a consequence of ZFC + AF) and the Foundation Axiom. Show, using reflection

$$(\exists V)[(\forall x \in V)(\exists y \in V)[x \in y] \wedge (\exists x)[x \in V]]$$

(hint: show first $(\forall x)(\exists y)[x \in y]$ which is not obvious without replacement) and hence the infinity axiom can be deduced from the reflection principle. (One can derive also the replacement axiom from the reflection principle.)

Constructible Sets

Do we have $L_n = V_n$ $(n \in \omega)$? $L_\omega = V_\omega$? $L_{\omega+1} = V_{\omega+1}$? Explain why.

We know that $\omega \in L_{\omega+1}$ and that there exists an ordinal α such that

$$L(a) \wedge a \subseteq \omega \rightarrow a \in L_\alpha$$

What is the smallest possible value of α??