AXIOMATIC SET THEORY 2

Ordinal Arithmetic

Recall that a *well-ordered set* is an ordered set $(X, \leq)$ such that any non empty subset of X has a least element. Explain why, given any such X there exists one and only one ordinal α such that $(X, <)$ is isomorphic to (α, ϵ).

Explain why we can define the operation $+$ on ordinals such that

$$\alpha + 0 = \alpha, \quad \alpha + (\beta + 1) = (\alpha + \beta) + 1, \quad \alpha + \lambda = sup_{\xi < \lambda}(\alpha + \xi)$$

for limit λ, and then the operation $\cdot$

$$\alpha \cdot 0 = 0, \quad \alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha, \quad \alpha \cdot \lambda = sup_{\xi < \lambda}(\alpha \cdot \xi)$$

for limit λ.

If $(X, <)$ is isomorphic to α and $(Y, <)$ isomorphic to β notice that $\alpha + \beta$ is isomorphic to the disjoint $X + Y$ with the elements of Y are $>$ the elements of X and $\alpha \cdot \beta$ is isomorphic to the lexicographic ordering $X \times Y$.

Give an example where $\alpha + \beta \neq \beta + \alpha$.

Show that $\mu + \alpha = \mu + \beta$ implies $\alpha = \beta$ and that $\mu \cdot \alpha = \mu \cdot \beta$ implies $\alpha = \beta$ if $0 < \mu$.

If $\alpha \leq \beta$ show that there is exactly one ξ such that $\alpha + \xi = \beta$.

If $\mu < \alpha \cdot \beta$ show that there exist two ordinals α_1, β_1, uniquely determined, such that $\alpha_1 < \alpha$, $\beta_1 < \beta$ and $\mu = \alpha\beta_1 + \alpha_1$.

An ordinal $\rho > 0$ is said to be *indecomposable* iff there exist no pair of ordinals $\alpha, \beta < \rho$ such that $\rho = \alpha + \beta$.

Explain why ω is indecomposable. What is the least indecomposable ordinal?

Show that ρ is indecomposable iff $\alpha + \rho = \rho$ for each $\alpha < \rho$.

Let α be an ordinal > 0 show that there exists a greatest indecomposable ordinal among the indecomposable ordinal $\leq \alpha$. Deduce from this that the least upper bound of a set of indecomposable ordinals is indecomposable.

Show that if $0 < \alpha$ then $\alpha \cdot \omega$ is the least ordinal $> \alpha$ which is indecomposable.

We define now

$$\alpha^0 = 1, \quad \alpha^{\beta+1} = (\alpha^\beta) \cdot \alpha, \quad \alpha^\lambda = sup_{\xi < \lambda}(\alpha^\xi)$$

for limit λ.

What is 2^ω?

Show that an ordinal is indecomposable iff it is of the form ω^β.