AXIOMATIC SET THEORY 1

Cantor-Bendixson Analysis

The goal is to show that the Continuum Hypothesis holds in ZFC for *closed* subset of the real line R. Given $X \subseteq R$ we define the *derivative* X' of X to be the set of all non isolated points of X: an element $x \in X$ is in X' iff any open interval containing x contains also another point in X. If X is closed show that $x \in X'$ iff for any open interval I containing x the intersection $I \cap X$ is infinite. Show that if X is closed, then $X' \subseteq X$ is also closed.

We say that X is *perfect* iff $X = X'$. Show that if X is perfect and non empty then it has $2^{\aleph_0}$ elements (hint: build a sequence I_σ of intervals indexed by binary sequence meeting X such that $I_{\sigma 0} \cap I_{\sigma 1} = \emptyset$.)

Starting from a closed subset $X \subseteq R$ define by induction a decreasing sequence of closed subsets $X^{(\alpha)}$

$$X^{(0)} = X, \qquad X^{(\alpha+1)} = (X^{\alpha})', \qquad X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$$

for λ limit. Historically, ordinals were introduced by Cantor precisely in order to define such a sequence.

Explain why there should exist $\alpha < \aleph_1$ such that $X^{(\alpha)} = X^{(\alpha+1)}$. (Hint: to any α such that $X^{(\alpha)} \neq X^{(\alpha+1)}$ associate a rational interval which meets $X^{(\alpha)}$ but not $X^{(\alpha+1)}$.) Explain also why each $Y - Y'$ is at most countable. (Hint: each point in $Y - Y'$ is isolated by an open rational interval.)

Conclude from this that if $X \subseteq R$ is closed it is either at most countable or of cardinality $2^{\aleph_0}$.

König's theorem

The goal is to show in ZFC that we cannot have $\overline{\overline{R}} = \aleph_\omega$.

Let $(A_i), (B_i), i \in I$ be two families of sets such that $\overline{\overline{A_i}} < \overline{\overline{B_i}}$ for all i. Show that $\overline{\overline{\Sigma A_i}} < \overline{\overline{\Pi B_i}}$. (Hint: given $f : \Sigma A_i \to \Pi B_i$ build $(b_i) \in \Pi B_i$ which is not in the image of f.) Conclude by taking $A_i = \aleph_i,\ B_i = R,\ i \in \omega$.

Cantor-Bernstein Theorem

The goal of this exercice is to show, *without* using the Axiom of Choice, that if we have an injection $f : A \to B$ and an injection $g : B \to A$ then we have a bijection between A and B.

Explain first how to prove this using AC and the notion of cardinal of a set.

Next, remark that we can suppose A, B disjoint and consider f, g as defining a bipartite graph over $A \cup B$ where the edges are $(x, f(x)),\ x \in A$ and $(y, g(y)),\ y \in B$. Conclude by considering the connected component of this graph.