COMPACT SPACES AND DISTRIBUTIVE LATTICES

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Introduction

This note presents a general construction (theorem 2.1) connecting compact locales and distributive lattices. This allows us to reduce results about compactness of locales to theorems about distributive lattices. We give two applications: a reduction of the localic version of Tychonoff's theorem [7], that states that a product of compact locales is locales, to a result about coproduct of distributive lattices, and a proof of the localic version of Steenrod's theorem, that states that an inverse limits of compact locales is compact. Our proof of Tychonoff's theorem does not use any decidability hypotheses on the index set[1]. The localic version of Steenrod's theorem appears in [6], but with a proof that uses classical logic. One noteworthy feature of our arguments is that they can be formulated both in topos theory and in a predicative theory such as CZF [1, 2].

Another method to analyse compactness is provided by the theory of preframes [3, 9, 11], that are structures closed under finite meets and *directed* joins (thus, any frame defines canonically a preframe). For instance, Tychonoff's theorem is a direct consequence of the remarkable fact that a preframe coproduct of a family of frames, seen as preframes, is actually a frame (which is the frame coproduct of this family). It seems likely that the theory of preframes would provide as well a constructive proof of the localic Steenrod's theorem and it may actually be interesting to formulate this theory in a predicative framework.

1. Formal Spaces and Distributive Lattices

It will be convenient here to work with the following modification of the notion of site [8]. We represent a locale (or formal space) X as a distributive lattice D together with a nucleus $j : Idl(D) \to Idl(D)$ on the frame $Idl(D)$ of ideals of the lattice D. To give such a nucleus is equivalent to give a *covering relation* on D that is a relation $a \lhd U$ between elements of D and ideals of D satisfying

- $a \lhd U$ if $a \in U$
- $a \lhd U_1 \cap U_2$ if $a \lhd U_1$ and $a \lhd U_2$
- $a \lhd V$ if $a \lhd U$ and $u \lhd V$ for all $u \in U$
- $a \lhd U$ if $a \leq b$ and $b \lhd U$
- $0 \lhd U$
- $a_1 \vee a_2 \lhd U$ if $a_1 \lhd U$ and $a_2 \lhd U$

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[1] This general version of Tychonoff's theorem appears in [12], but we believe that our argument is simpler.

We write $a \lhd b$ for $a \lhd \{x \in D \mid x \leq b\}$. Also, if U is an arbitrary subset of D we write $a \lhd U$ for $a \lhd U'$ where U' is the ideal of D generated by U.

The lattice $Idl(D, \lhd)$ of *j-closed* ideals U, ideals such that $j(U) = U$ forms a frame, and hence defines a locale $Sp(D, \lhd)$. If j is the identity nucleus, we write simply $Sp(D)$ which is the usual spectral locale associated to D. If X is the locale $Sp(D, \lhd)$ its associated frame of formal opens $O(X)$ is thus $Idl(D, \lhd)$.

If X, Y are two locales, to give a continuous map $f : Y \to X$ is to give a frame map $f^* : O(X) \to O(Y)$ (preserving finite meets and arbitrary sups). If X is $Sp(D, \lhd)$ to give such a map is equivalent to give a lattice map $f^* : D \to O(Y)$ such that

$$a \lhd U \to f^*(a) \leq \bigvee_{u \in U} f^*(u)$$

By considering the distributive lattice generated by basic open sets it is clear that the following representation result holds.

Proposition 1.1. *Any locale X can be represented of the form $Sp(D, \lhd)$ where furthermore $a \lhd b$ iff $a \leq b$ in D.*

In a predicative framework such as CZF, what is important is that while the frame $O(X)$ is a class, we can furthermore choose D to be a set [2].

2. Compact topologies

Let D be an arbitrary distributive lattice. We define a relation $a \prec U$ between elements of D and ideals of U by

$$\forall x. a \vee x = 1 \to \exists u \in U. u \vee x = 1$$

We write $j_C(U)$ the set $\{a \in D \mid a \prec U\}$.

Theorem 2.1. *The function j_C is a nucleus on the frame of ideals of D. Furthermore the locale $Sp(D, \prec)$ is compact. More generally, if $\lhd$ is any covering relation such that $\lhd \subseteq \prec$ then $Sp(D, \lhd)$ is compact.*

Proof. We check that $\prec$ is a covering relation.

- $a \prec U$ if $a \in U$: this is direct;
- $a \prec U_1 \cap U_2$ if $a \prec U_1$ and $a \prec U_2$: assume $a \vee x = 1$ then there exists $u_i \in U_i$ such that $x \vee u_i = 1$ for $i = 1, 2$ and we have $x \vee u_1 u_2 = 1$ and $u_1 u_2 \in U_1 \cap U_2$;
- $a \prec V$ if $a \prec U$ and $u \prec V$ for all $u \in U$: assume $a \vee x = 1$ then there exists $u \in U$ such that $u \vee x = 1$ and since $u \prec V$ there exists $v \in V$ such that $v \vee x = 1$ as desired;
- $a \prec U$ if $a \leq b$ and $b \leq U$: assume $a \vee x = 1$ then we have $b \vee x = 1$ and thus there exists $u \in U$ such that $u \vee x = 1$;
- $0 \prec U$ is direct;
- $a_1 \vee a_2 \prec U$ if $a_1 \prec U$ and $a_2 \prec U$: assume $a_1 \vee a_2 \vee x = 1$, there exists $u_1 \in U$ such that $a_2 \vee u_1 \vee x = 1$ and then there exists $u_2 \in U$ such that $(u_2 \vee u_1) \vee x = 1$. This shows that there exists $u = u_1 \vee u_2 \in U$ such that $u \vee x = 1$.

Assume now $\lhd \subseteq \prec$. If $1 \lhd U$ then $1 \prec U$ which implies $1 \in U$. This shows that $Sp(D, \lhd)$ is compact. □

The next proposition expresses that in some sense, $\prec$ is the greatest compact covering relation on D.

Proposition 2.2. *If $\lhd$ is a compact covering relation on D such that $a \lhd b$ iff $a \leq b$ then $\lhd \subseteq \prec$.*

Proof. Assume $a \lhd U$. If $a \vee x = 1$ there exists, since $Sp(D, \lhd)$ is compact, $u \in U$ such that $1 \lhd u \vee x$. By hypothesis on $\lhd$ this is equivalent to $1 = u \vee x$. Thus we have $a \prec U$. □

3. Tychonoff's theorem

The following proposition has a direct proof.

Proposition 3.1. *Let $Sp(D_i, \lhd_i)$ a family of compact locales, and let $D, \sigma_i : D_i \to D$ be the coproduct of the distributive lattices D_i. The product of the locales $Sp(D_i, \lhd_i)$ is the locale $Sp(D, \lhd)$ where $\lhd$ is the least covering relation on D such that*

$$a \lhd_i U \to \sigma_i(a) \lhd \sigma_i(U)$$

We remark that in predicative systems such as CZF, we have to rely on some further assumptions on the covering relations $\lhd_i$ (namely that they are *set-based* [2]) to ensure the existence of such a covering relation $\lhd$.

Using the representation results above, we have thus reduced Tychonoff's theorem to the following theorem about distributive lattices. Let D_i be a family of distributive lattices and $D, \sigma_i : D_i \to D$ its coproduct.

Theorem 3.2. *If $a \in D_i$ and U is an ideal of D_i*

$$a \prec U \to \sigma_i(a) \prec \sigma_i(U)$$

This theorem states that we have continuous projection maps $Sp(D, \prec) \to Sp(D_i, \prec)$. To prove this theorem, we first formulate some general lemmas. Let S be the set $(\Sigma i \in I)D_i$ and $\phi : S \to D$ be the funtion $\phi(i, a) = \sigma_i(a)$.

Lemma 3.3. *If A, B are two finite subsets of S such that*

$$\wedge_{a \in A} \phi(a) \leq \vee_{b \in B} \phi(b)$$

then there exists $i \in I$ and two finite subsets X, Y of D_i such that

$$\wedge_{x \in X} x \leq \vee_{y \in Y} y$$

in D_i and $(i, x) \in A$ for $x \in X$ and $(i, y) \in B$ for $y \in Y$.

Proof. This follows for instance from the method presented in [5]. □

Lemma 3.4. *If $a \in D_i$ and $\sigma_i(a) \vee x = 1$ in D then there exists $y \in D_i$ such that $a \vee y = 1$ in D_i and $\sigma_i(y) \leq x$.*

Proof. We can assume x to be a disjunction of elements of the form $\sigma_j(b)$ in which case this follows from lemma 3.3 (with $A = \emptyset$). □

We can now prove the theorem.

Proof. Assume $a \prec U$ in D_i. If $\sigma_i(a) \vee x = 1$ in D then by lemma 3.4 we can find $y \in D_i$ such that $a \vee y = 1$ in D_i and $\sigma_i(y) \leq x$. Since $a \prec U$ we can find $u \in U$ such that $u \vee y = 1$. We have then $\sigma_i(u) \in \sigma_i(U)$ such that $x \vee \sigma_i(u) = 1$. □

Corollary 3.5. *(Tychonoff's theorem) The product of a family of compact locales is compact.*

Proof. By proposition 2.2 we can assume that the family is of the form $X_i = Sp(D_i, \lhd_i)$ with $\lhd_i \subseteq \prec$. The product of X_i is then of the form $Sp(D, \lhd)$ where $D, \sigma_i : D_i \to D$ is the coproduct of the distributive lattices D_i and $\lhd$ is the least covering relation on D such that

$$a \lhd_i U \to \sigma_i(a) \lhd \sigma_i(U)$$

By theorem 3.2, we have $\lhd \subseteq \prec$ and hence by theorem 2.1, $Sp(D, \lhd)$ is compact. □

This proof can be seen as a simplification of the proof in [4].

4. Steenrod's Theorem

We assume now given an inverse system

$$f_{ij} : Sp(D_j, \lhd_j) \to Sp(D_i, \lhd_i)$$

where each $\lhd_i$ is a compact covering on the lattice D_i such that $a \lhd b$ iff $a \leq b$ in D_i. For $i \leq j$ we have a lattice map $f_{ij}^* : D_i \to Idl(D_j, \lhd_j)$ such that

$$a \lhd_i U \to f_{ij}^*(a) \lhd_j f_{ij}^*(U)$$

We then let S be $(\Sigma i \in I) D_i$ and $\vdash$ be the entailment relation [5] on S defined by

$$(i_1, a_1), \ldots, (i_n, a_n) \vdash (j_1, b_1), \ldots, (j_m, b_m)$$

iff there exists $k \geq i_1, \ldots, i_n, j_1, \ldots, j_m$ such that

$$f_{i_1 k}^*(a_1) \wedge \cdots \wedge f_{i_n k}^*(a_n) \leq f_{j_1 k}^*(b_1) \vee \cdots \vee f_{j_m k}^*(b_m)$$

in the frame $Idl(D_k, \lhd_k)$. We let $D, m : S \to D$ be the distributive lattice generated by this entailment relation [5] and $\sigma_i : D_i \to D$ be the lattice map $\sigma_i(a) = m(i, a)$.

Proposition 4.1. *The inverse limit of the system $Sp(D_i, \lhd_i), (f_{ij})$ can be represented as the space $Sp(D, \lhd)$ where $\lhd$ is the least relation on D such that*

$$a \lhd_i U \to \sigma_i(a) \lhd \sigma_i(U)$$

Theorem 4.2. *The inverse limit $Sp(D, \lhd)$ is compact.*

Proof. As before, we are reduced to prove that if $a \in D_i$ and $U \in Idl(D_i, \lhd_i)$ then

$$a \lhd_i U \to \sigma_i(a) \prec \sigma_i(U)$$

Assume $a \lhd_i U$ and $x \vee \sigma_i(a) = 1$. We can assume x to be of the form $\sigma_{i_1}(a_1) \vee \cdots \vee \sigma_{i_n}(a_n)$ and we can then find $k \geq i, i_1, \ldots, i_n$ such that

$$1 = f_{ik}^*(a) \vee f_{i_1 k}^*(a_1) \vee \cdots \vee f_{i_n k}^*(a_n)$$

in the frame $Idl(D_k, \lhd_k)$. Since $a \lhd_i U$ we have $f^*_{ik}(a) \lhd_k f^*_{ik}(U)$ and hence
$$1 = f^*_{ik}(U_i) \vee f^*_{i_1 k}(a_1) \vee \cdots \vee f^*_{i_n k}(a_n)$$
Since the frame $Idl(D_k, \lhd_k)$ is compact, we can find
$$u \in f^*_{ik}(U),\ \ u_1 \in f^*_{i_1 k}(a_1), \ldots, u_n \in f^*_{i_n k}(a_n)$$
such that
$$1 \lhd_k u \vee u_1 \vee \cdots \vee u_n$$
and hence
$$1 = u \vee u_1 \vee \cdots \vee u_n$$
This implies $\sigma_k(u) \vee x = 1$ and $\sigma_k(u)$ is in the ideal $\sigma_i(U)$. $\square$

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