Sequents, Frames, and Completeness

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Abstract. Entailment relations, originated from Scott, have been used for describing mathematical concepts constructively and for representing categories of domains. This paper gives an analysis of the freely generated frames from entailment relations. This way, we obtain completeness results under the unifying principle of the spatiality of coherence logic. In particular, the domain of disjunctive states, derived from the hyperresolution rule as used in disjunctive logic programs, can be seen as the frame freely generated from the *opposite* of a sequent structure. At the categorical level, we present equivalences among the categories of sequent structures, distributive lattices, and spectral locales using appropriate morphisms. Near the end of the paper a representation of stably compact frames is given as an application of this approach.

Introduction

Entailment relations were introduced by Scott as an abstract description of Gentzen's sequent calculus [17–19]. It can be seen as a generalisation of the earlier consequence calculus of Hertz [11] to a *multi-conclusion* consequence relation. The notion of consequence relation, with only one conclusion, has been analysed by Tarski [22]. This consequence calculus has been used by Scott in order to give a concrete representation of domains, as in information systems [20]. It is thus natural to wonder if the more general notion of entailment relation, with multiple conclusions, can be used to represent larger categories of domains, such as those related to non-determinism. This is indeed the case, and it has been developed in [23, 24] and [6], in an independent way from Scott's work on entailment relations (in [23], a set together with an entailment relation is called a *sequent structure*). Another related reference, also independent from Scott's work, is [10].

In this paper we analyse various completeness theorems for sequent structures by embedding them into frames. A goal of this study is to provide a unified

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way to present completeness results in logic, such as those for resolution and hyperresolution.

A number of recent developments serve as the motivation for the current paper. In [4, 5], it is shown that entailment relations are naturally connected to several mathematical structures. They can be used to give elegant constructive version of some basic mathematical concepts (and theorems), such as continuous linear forms, space of valuations, etc. One key point here is that it is often possible to get direct explicit descriptions of entailment relations generated by some rules, avoiding syntactical induction and case analysis on derivations.

In order to understand appropriate domains for the semantics of disjunctive logic programs, [25] introduces *clausal logic* based on the so-called hyperresolution rule [14]. Completeness of hyperresolution provides the basis for this domain-theoretic semantics: it establishes the equivalence of the model-theoretic semantics and the proof-theoretic semantics. Here, a set of clauses closed under hyperresolution is called a disjunctive state; the collection of disjunctive states under inclusion forms a complete lattice, which, in the case of information systems, is isomorphic to the Smyth powerdomain [15, 25].

A natural question is whether the cpo of disjunctive states can be seen as a universal construction for sequent structures. Related to this question is the canonical embedding of a sequent structure into a frame. For this purpose we use Johnstone's coverage method [9] to study frames generated from a sequent structure as well as from its opposite. Interestingly, the frame generated from the opposite is precisely the complete lattice of disjunctive states. Moreover, in each case the universal map gives a way to capture a point of the frame as an ideal element of the underlying sequent structure.

The completeness theorem of coherent logic states that any coherent (or spectral) frame is *spatial* [9]. It ensures that enough models exist to uniquely determine the partial order, where models correspond to completely prime filters. This means that when sequent structures are embedded into spectral frames, we have enough models to uniquely determine the entailment relation, and thus obtain certain completeness result "for free", such as the completeness of hyperresolution. In return, existing results [15, 25] related to hyperresolution suggest several explicit constructions for the sequent-structure-generated frames: a semantical one, a proof-theoretic one, and a third one based on the notion of "choice inference".

A couple of results in this paper may be seen as "folklore"; their roots may be traced back eventually to Stone's representation theorem [21]. We feel however that our contribution lies in tying in the more discrete notion of sequent structures with the more complete notion of locales through the so-called coverage relation [9] in a concrete logical setting. This allows the importation of existing results in locales to sequent structures, shedding new light on the topic. It is, for instance, quite interesting that the hyperresolution rule appears naturally in solving the problem of embedding an entailment relation in a frame, and it may not be obvious a priori that the disjunctive states form a frame. We hope that this paper is a first step in exploring completeness of various logical systems by means of canonical embedding to locales.

1 Frames and coverage

A *frame* is a poset with finite meets and arbitrary joins which satisfies the infinite distributive law

$$x \wedge \bigvee U = \bigvee \{x \wedge y \mid y \in U\}.$$

For frames F and G, a frame morphism is a function $f : F \to G$ that preserves finite meet and arbitrary joins. Frames are also called *locales.*

Johnstone ([9], page 57) provides a way to construct a frame from a meet-semi-lattice based on the notion of *coverage relation.*

Definition 1. *Let $(S, \wedge, \leq)$ be a meet-semi-lattice. A* coverage *on S is a relation $\succ \subseteq \mathcal{P}S \times S$ between subsets and elements of S satisfying*

1. *if $U \succ a$ then for any $y \in U$, $y \leq a$.*
2. *if $U \succ a$ then for any $b \leq a$, $\{y \wedge b \mid y \in U\} \succ b$.*

A $\succ$-ideal determined by coverage $\succ$ is a subset I of S which is

1. *lower-closed: $a \in I$ & $b \leq a \Rightarrow b \in I$,*
2. *covered: $U \succ a$ & $U \subseteq I \Rightarrow a \in I$.*

A meet-semi-lattice S equipped with a coverage relation $\succ$ is called a *site.* A frame H with $i : S \to H$ is said to be *generated* from a site $(S, \succ)$ if

- i preserves finite meets,
- i transforms covers to joins: $U \succ a \Rightarrow i(a) = \bigvee i(U)$, and
- H, i is universal, i.e., the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & F \\ H \arrow[ur, "\exists ! g"'] & \end{tikzcd}$$

Remark. For here and for the rest of the paper, all maps are assumed to preserve the respective structures they are acting on. This remark will be implicitly in force for all commutative diagrams and will not be repeated. We also remark in general that such a universal property guarantees that the generated structures are always *unique up to isomorphism.*

Here is Johnstone's basic result for the coverage relation.

Theorem 1 (Coverage Theorem [9], page 58). *The collection of $\succ$-ideals under inclusion is the frame, written $\mathsf{Frm}(S, \succ)$, generated from a site $(S, \succ)$.*

We write $\mathsf{Frm}(S)$ if $\succ$ is clear from the context.

We do not recall the proof of this result. However, we mention a key lemma in this proof. For $U \subseteq S$, $\mathsf{c}U$ stands for the least $\succ$-ideals containing U.

Lemma 1. *If $U, V \subseteq S$ are lower-closed then $\mathsf{c}(U \cap V) = \mathsf{c}U \cap \mathsf{c}V$.*

Recall that a frame can be seen as a "point-free" description of the open sets of a topological space. In this view, points are not basic, but are defined as collection of opens: a *point* of a frame is a completely prime filter, i.e. a filter α such that if $\bigvee U \in \alpha$ then there exists $x \in U$ such that $x \in \alpha$. If H is generated from $(S, \succ)$ then a point is determined by its restriction to S, which is a filter α of S such that $a \in \alpha \;\&\; U \succ a \;\Rightarrow\; (\exists b \in U)\; b \in \alpha$.

A frame H is called *spatial* (or has *enough points*) if for any $a, b \in H$, $a \leq b$ iff $\forall \alpha,\ a \in \alpha$ implies $b \in \alpha$, where α ranges over points of H. Intuitively, if we regard a, b as sets of points, then $a \leq b$ exactly when $a \subseteq b$.

2 Spectral frames

There is a standard way to generate a frame from a distributive lattice D. One defines the coverage by letting $U \succ a$ if and only if $U \subseteq \downarrow a$ and there exists a finite subset X of U such that $a = \vee X$. By distributivity, this is a coverage relation. A $\succ$-ideal is then exactly an *ideal* of D: a downward-closed subset of D closed under finite joins. The generated frame is precisely the so-called *ideal completion* of D, which is written as $\mathsf{Idl}(D)$.

Definition 2. *A frame (locale) is* coherent *or* spectral *if it is isomorphic to the ideal completion of a distributive lattice*[1].

Let **Spec** be the category of spectral frames and frame morphisms, and **Lat** be the category of distributive lattices. Any frame H can be seen as a distributive lattice $\mathsf{F}(H)$, which defines a functor $\mathsf{F} : \mathbf{Spec} \to \mathbf{Lat}$. We call a map $m : D \to \mathsf{F}(H)$ an *interpretation* of the distributive lattice D into the frame H. If D is equipped with a coverage relation described above, a map $i : D \to H$ preserving finite meets and transformings covers to joins is exactly an interpretation of D into H. In such a case the map $i : D \to \mathsf{Idl}(D)$ is universal among the interpretation $f : D \to H$. Another way to say this is that $\mathsf{Idl} : \mathbf{Lat} \to \mathbf{Spec}$ is a left-adjoint to the functor F.

Let **RelLat** be the category of distributive lattices and maps $m : D \to \mathsf{Idl}(E)$.

Theorem 2. *The categories* **RelLat**, **Spec** *are equivalent.*

Proof. This follows from [13], Chapter VI, 5, Exercise 2 and the remarks made above. ⊠

[1] The term *coherent* is used in such a way in [9]. But it is used with another meaning in domain theory or even in [10]. The term *spectral*, used because such frames are exactly the ones that are spectrum of a commutative ring, is less ambiguous.

Another important corollary of the ideal completion construction is the following representation theorem for distributive lattices in localic form.

Theorem 3. *If $i : D \to H$ is an universal interpretation then $x \leq y$ iff $i(x) \leq i(y)$.*

Proof. Indeed, it is enough to look at the map $i : D \to \mathsf{Idl}(D)$ where $i(x)$ is the ideal generated by x. We have then $i(x) = \downarrow x$ and if $i(x) \leq i(y)$ then $x \in i(y)$ and so $x \leq y$. $\boxtimes$

We give now a useful "finite" version of Johnstone's coverage theorem. A site $(S, \succ)$ is called *finite* if $\succ \subseteq \mathrm{Fin}S \times S$ is a relation between finite subsets of S and elements of S. A finitely generated $\succ$-ideal is the $\succ$-ideal $\mathsf{c}U$ generated from some finite subset U of S.

A distribitive lattice L with $i : S \to L$ is said to be *generated* from a finite site $(S, \succ)$ if

- i preserves finite meets,
- i transforms covers to (finite) joins: $Y \succ a \Rightarrow i(a) = \bigvee i(Y)$, and
- L, i is universal, i.e., the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & D \\ L \arrow[ur, "\exists ! g"'] & \end{tikzcd}$$

Theorem 4 (Finite Coverage Theorem). *For a finite site $(S, \succ)$, the collection of finitely generated $\succ$-ideals under inclusion form a distributive lattice, written $\mathsf{L}(S, \succ)$. We write $\mathsf{L}(S)$ if $\succ$ is clear from the context.*

Proof. If $X, Y \subseteq S$ are finite then

$$\downarrow X \cap \downarrow Y = \downarrow \{x \wedge y \mid x \in X, y \in Y\}.$$

By Lemma 1,

$$\mathsf{c}X \cap \mathsf{c}Y = \mathsf{c}(\{x \wedge y \mid x \in X, y \in Y\}).$$

Hence, the collection of finitely generated $\succ$-ideals under inclusion is the sub-distributive lattice of $\mathsf{Frm}(S)$ generated by $i(x), x \in S$. The verification of the universal property is straightforward. $\boxtimes$

By universal properties, this theorem has the following consequence.

Theorem 5. *If $(S, \succ)$ is a finite site then $\mathsf{Frm}(S)$ is a spectral frame, which is the ideal completion of the lattice $\mathsf{L}(S)$ mentioned above.*

The following fact will be used in the rest of the paper.

Theorem 6 (Page 65, [9]). *Spectral frames are spatial.*

3 Sequent structures

Definition 3. *A* theory *T on a set A is a relation on finite subsets* $\mathrm{Fin}(A)$ *of A.*

Entailment relations, introduced by Scott in [17], are theories with some basic restrictions. If (X, Y) belongs to a theory T, this is now written $X \vdash Y$, with the understanding that the set X is considered as a conjunction and Y as a disjunction.

Definition 4. *An entailment relation (or a sequent structure) is a set A with a theory $\vdash$ such that*

$$
\begin{array}{ll}
(I) & a \vdash a \\[1ex]
(W) & \dfrac{X' \supseteq X \qquad X \vdash Y \qquad Y \subseteq Y'}{X' \vdash Y'} \\[2ex]
(C) & \dfrac{X \vdash Y,\, a \qquad a,\, X \vdash Y}{X \vdash Y}
\end{array}
$$

We use the notations $X, Y, \ldots$ for finite subsets of A, and X, Y for $X \cup Y$ while X, a for $X \cup \{a\}$.

Several properties of entailment relations are self-evident.

First, entailment relations are completely symmetric: $(A, \vdash)$ is an entailment relation iff $(A, \dashv)$ is.

Second, entailment relations are closed under arbitrary intersections.

Third, the largest relation on $\mathrm{Fin}(A)$ is an entailment relation. Given any theory T over A, the entailment relation $\vdash_T$ *generated by* T can be seen to be the intersection of all entailment relations on A containing T. One can also describe $\vdash_T$ as the closure of T under $(I), (W)$, and (C).

Last, information systems [20] can be seen as a special kind of entailment relation generated by rules of the form $X \vdash Y$ with Y being either a singleton or empty (intuitionistically they are more complex, and we need further a primitive notion of *consistent* sets of token).

Our intuition about the logical status of $\vdash$ is made precise be the following definition.

Definition 5. *For a distributive lattice D and a theory T on a set A, a map $i : A \to D$ is said to* model *T if $X \vdash Y \in T$ implies $\wedge i(X) \leq \vee i(Y)$. We say that the distributive lattice $\mathsf{L}(T)$ is* generated by *T if there is a model map $i : A \to \mathsf{L}(T)$ which is universal among all such maps:*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & L \\
\mathsf{L}(T) \arrow[ur, "\exists! g"'] &
\end{tikzcd}
$$

We can study the similar topic of interpreting a theory in a *frame*.

Definition 6. *Let H be a frame. An* interpretation *of a theory T on a set A in H is a map $m : A \to H$ such that if $X \vdash Y \in T$ then $\wedge m(X) \leq \vee m(Y)$. A frame* $\mathsf{Frm}(T)$ *is* generated *by T if there is a universal interpretation $m_0 : A \to \mathsf{Frm}(T)$:*

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "m_0"'] & H \\
\mathsf{Frm}(T) \arrow[ur, "\exists ! f"'] &
\end{tikzcd}$$

By universal property, it is clear that we have $\mathsf{Frm}(T) = \mathsf{Idl}(\mathsf{L}(T))$.

Finally, we recall what is an ideal element. Ideal elements have been used for representing domains. Given a theory T, the set of all of its ideal elements forms a dcpo under inclusion. One can obtain different categories of domains by considering different (sub)classes of sequent structures [24].

Definition 7. *A subset $x \subseteq A$ is called an ideal element with respect to a theory T on a set A if it is closed under entailment, that is, for any finite $X \subseteq A$,*

$$X \subseteq x \;\&\; (X, Y) \in T \;\Rightarrow\; x \cap Y \textit{ inhabited.}$$

Notice that $x \subseteq A$ is an ideal element of T iff it is an ideal element of $\vdash_T$.

A *co-element* of a theory T, seen as a relation on $\mathsf{Fin}(A)$, is an ideal element for the converse of T, that is the theory of sequents $Y \vdash X$ for $X \vdash Y \in T$. By logical transposition, one easily checks classically that y is a co-element of T iff y is the complement of an ideal element x of T; but our definition of co-element is formulated in a purely positive way.

4 Clausal logic and hyperresolution

The notion of clause is a basic concept in logic programming. A natural framework for reasoning about clauses, called *clausal logic*, is demonstrated in [15, 25] to play a fundamental role in disjunctive logic programming semantics.

With respect to a theory T on a set of token A, a *clause* is a finite subset of A, and a *clause set* is a collection of clauses. An ideal element x is a *model* of a clause u if $x \cap u \neq \emptyset$. x is a model of a clause set W if it is an model of every clause in W. There are three distinct notions of inference in clausal logic: $\models$, $\vdash^*_{hr}$, and the "choice inference" $\dashrightarrow$. For a clause set W and a clause u, we write

1. $W \models u$ if every model of W is a model of u. This is a model-theoretic concept, capturing the semantics.

2. $W \vdash_{hr}^{*} u$ if either $\emptyset \in W$, or u can be deduced from W using the so-called *hyperresolution rule*

$$\frac{a_1, X_1 \quad \ldots \quad a_n, X_n \quad a_1, \ldots, a_n \vdash_T Y}{X_1, \ldots, X_n, Y}$$

   This is clearly a proof-theoretic, or operational, concept.
3. $\{X_1, \ldots, X_n\} \dashrightarrow u$ if $\{a_i \mid 1 \leq i \leq n\} \vdash_T u$ for any choice $a_1 \in X_1, a_2 \in X_2, \ldots, a_n \in X_n$. This is an intermediate notion: it uses the notion of arbitrary choice.

A result of [25] is that the three distinct notions of inference are equivalent to each other.

Theorem 7 (Rounds and Zhang). *Let $(A, \vdash)$ be a sequent structure. Let W be a finite clause set, and u a clause. The following three items are equivalent:*

1. $W \models u$,
2. $W \vdash_{hr}^{*} u$,
3. $W \dashrightarrow u$.

For any clause set C, we write $\Uparrow C$ for the least clause set containing C and closed under hyperresolution. A *disjunctive state* is a clause set C such that $C = \Uparrow C$.

The concept of disjunctive state is well-behaved on sequent structures [25]:

Theorem 8. *For a sequent structure $\underline{A}$, the set of all its disjunctive states under inclusion is a complete lattice.*

This result will be refined, and the lattice of disjunctive state will be caracterised by an universal property.

5 Explicit construction of generated frame using coverage

It is possible to give an explicit construction of the generated frame from a theory T on a set A through an appropriate coverage relation defined by a dual form of hyperresolution.

For T be a theory on a set A, consider the meet-semi-lattice $(\mathrm{Fin}(A), \cup, \supseteq)$ and the relation defined by

$$\{a_1, Z, \ \ldots, \ a_n, Z\} \succ Z$$

iff there is a rule $X \vdash a_1, \ldots, a_n \in T$ such that $X \subseteq Z$. Note that if $X \vdash \emptyset \in T$ and $X \subseteq Z$, then we have $\{\ \} \succ X$ (one can take this to be the $n = 0$ case). This is clearly a coverage relation, according to Definition 1. A $\succ$-ideal is, by definition, precisely a subset $U \subseteq \mathrm{Fin}(A)$ such that

– if $X \in U$ and $Y \supseteq X$, then $Y \in U$;

– if $\{a_1, Z, \ldots, a_n, Z\} \subseteq U$ and $X \vdash a_1, \ldots, a_n \in T$, and $X \subseteq Z$ then $Z \in U$.

We call such $\succ$-ideals *conjunctive states* and write H_0 for the set of all conjunctive states. For a set $U \subseteq \mathrm{Fin}(A)$, we write $\mathsf{c}U$ for the conjunctive state generated by U.

There is a useful proof-theoretic reading of the generated conjunctive state. For any set $U \subseteq \mathrm{Fin}(A)$, its generated state $\mathsf{c}U$ consists of all Xs that can be derive from assumptions from U by using the two rules of inference:

$$(M) \qquad \frac{Y}{X} \quad if\ Y \subseteq X$$

$$(H) \qquad \frac{a_1, Z \quad \ldots \quad a_n, Z}{Z} \qquad provided\ \ X \vdash a_1, \ldots, a_n \in T \quad X \subseteq Z$$

By Theorem 1, we immediately obtain that the set of conjunctive states under inclusion is the frame generated from the meet-semi-lattice $(\mathrm{Fin}(A), \cup, \supseteq)$ with coverage $\succ$, which depends on $\vdash$. We show that this frame has the required universal property for an interpretation.

Lemma 2. *Let H be any frame. There is a bijection between (finite) meet-preserving maps $i : \mathrm{Fin}(A) \to H$ that transforms covers to joins, and interpretations $m : A \to H$.*

Proof. Suppose $i : \mathrm{Fin}(A) \to H$ preserves finite meets and transforms covers to joins. Define a map $m_i : A \to H$ by letting $m_i(a) := i(\{a\})$ for each $a \in A$. We show that m_i is an interpretation. Since i preserves finite meets and meet for $\mathrm{Fin}(A)$ is set union, we have, for any finite $X \subseteq A$,

$$i(X) = i(\bigcup_{a \in X} \{a\}) = \wedge_{a \in X} i(\{a\}) = \wedge_{a \in X} m_i(a) = \wedge m_i(X).$$

Suppose $X \vdash Y$, with $Y = \{a_1, \ldots, a_n\}$. By the definition of $\succ$,

$$\{a_1, X, \ldots, a_n, X\} \succ X.$$

Since i transforms covers to joins, we have

$$\begin{aligned} i(X) &= i(X \cup \{a_1\}) \vee \cdots \vee i(X \cup \{a_n\}) \\ &\leq i(\{a_1\}) \vee \cdots \vee i(\{a_n\}) \\ &= \vee m_i(Y). \end{aligned}$$

Therefore, $\wedge m_i(X) \leq \vee m_i(Y)$, as needed. (Note that when $X \vdash \emptyset$, the empty collection $\{\ \}$ covers X, by definition. Transforming covers to joins in this case means $i(X) = \bigvee \emptyset = 0$, which can be restated as $\wedge m_i(X) \leq \vee m_i(\emptyset)$.)

Suppose, on the other hand, that $m : A \to H$ is an interpretation. We define a map $i_m : \mathrm{Fin}(A) \to H$ by letting $i_m(X) := \wedge m(X)$ for each $X \in \mathrm{Fin}(A)$. By this definition, i_m automatically preserves finite meets. We show that it also transforms covers to joins. If $\{a_1, X,\ a_2, X,\ \ldots,\ a_n, X\} \succ X$ then by definition $X \vdash a_1, \ldots, a_n$. Therefore,

$$\wedge m(X) \leq m(\{a_1\}) \vee m(\{a_2\}) \vee \cdots \vee m(\{a_n\}).$$

By distributivity, we have

$$\wedge m(X) = (\wedge m(X \cup \{a_1\})) \vee \cdots \vee (\wedge m(X \cup \{a_n\})).$$

This means $i_m(X) = i_m(X \cup \{a_1\}) \vee \cdots \vee i_m(X \cup \{a_n\})$, which is exactly the required property of "transforming covers to joins".

It is clear that the given transformations $i \longmapsto m_i$ and $m \longmapsto i_m$ amount to a bijection. ⊠

By the previous lemma and the Coverage Theorem, we arrive at the next conclusion, which says that H_0 is the generated frame from the theory T.

Theorem 9. *For any theory T on a set A, the set of its conjunctive states H_0 is a frame under inclusion. Moreover, the interpretation $m_0 : A \to H_0$ mapping a to $\mathsf{c}\{a\}$ is universal. Furthermore we have $X \vdash_T Y$ if and only if $\wedge m_0(X) \leq \vee m_0(Y)$.*

Proof. The last part, it suffices to show that $X \vdash_T Y$ whenever X can be proved from $\{\{y\} \mid y \in Y\}$ using the two rules of inference $(M), (H)$ above. For this, one checks that

- $X \vdash_T Y$ if $X' \subseteq X$ and $X' \vdash_T Y$, which follows from (W);
- $Z \vdash_T Y$ if $X \subseteq Z$ and $X \vdash a_1, \ldots, a_n \in T$ and $Z, a_i \vdash_T Y$ for all i, which follows from (W) and n applications of (C). ⊠

As a simple syntactical application of this result, we note:

Corollary 1. *If $X \vdash_T Y$ then we can derive X from $\{\{y\} \mid y \in Y\}$ using the two rules of inference $(M), (H)$.*

Proof. Indeed $m_0 : A \to H_0$ is an interpretation, and thus if $X \vdash_T Y$ we have $\wedge m_0(X) \leq \vee m_0(Y)$ in H_0 which means that we can derive X from $\{\{y\} \mid y \in Y\}$ using the two rules of inference $(M), (H)$. ⊠

Of course, one can derive this result in a purely combinatorial way.

By a similar argument, the following result follows as an application of the Finite Coverage Theorem (this result appears in [4] with a different proof).

Theorem 10. *For any theory T on a set A, the set of its finitely generated conjunctive states L_0 is a distributive lattice under inclusion. Moreover, the model map $m_0 : A \to H_0$ sending a to $\mathsf{c}\{a\}$ is universal. Furthermore we have $X \vdash_T Y$ if and only if $\wedge m_0(X) \leq \vee m_0(Y)$.*

As a result of Theorems 9 and 10, we can talk about joins and meets of finite subsets of A, with the understanding that such operations are always carried out in the generated frame (or in the generated lattice). *This is indeed the notational convention we adopt for the rest of the paper*: $\wedge X$ stands for $\wedge m_0(X)$.

6 Ideal elements, prime and completely prime filters

What makes the results given in the previous section useful is that we have a canonical correspondence between ideal elements of the sequent structure, prime filters of the distributive lattice, and completely prime filters of the generated frame. We establish the correspondence in this section.

Let T be a theory on A and $A \xrightarrow{i} \mathsf{L}(T)$, $A \xrightarrow{m} \mathsf{Frm}(T)$ be the universal maps for the generated distributive lattice $\mathsf{L}(A)$ and generated frame $\mathsf{Frm}(A)$, respectively. For $x \subseteq A$, define $I_x \subseteq \mathsf{L}(A)$ as

$$I_x := \{u \in \mathsf{L}(T) \mid (\exists X \subseteq x) \ \wedge X \leq u\}$$

and define $J_x \subseteq \mathsf{Frm}(T)$ in exactly the same way:

$$J_x := \{u \in \mathsf{Frm}(T) \mid (\exists X \subseteq x) \ \wedge X \leq u\}.$$

We have the following result, which shows that ideal elements, prime filters, and completely prime filters uniquely determine each other under their respective universal maps. By theorem 9 we can assume that $\mathsf{Frm}(T)$ is the lattice of conjunctive states of T.

Theorem 11. *1. If I is a prime filter of $\mathsf{L}(T)$ then the restriction of I to A, that is the set $i^{-1}(I)$, is an ideal element. Conversely if x is an ideal element of T then $I_x \subseteq \mathsf{L}(T)$ is a prime filter such that $x = i^{-1}(I_x)$.*

2. If J is a completely prime filter of $H_0 = \mathsf{Frm}(T)$ then the restriction of J to A, that is the set $m^{-1}(J)$, is an ideal element. Conversely if x is an ideal element of T then $J_x \subseteq \mathsf{Frm}(A)$ is a completely prime filter such that $x = m^{-1}(J_x)$.

Proof. We prove only the second item, since the arguments for both items are quite similar.

Suppose J is a complete prime filter of H_0. We show that $m_0^{-1}(J)$ is an ideal element. Suppose $X \vdash Y \in T$ and $X \subseteq m_0^{-1}(J)$. Then $m_0(X) \subseteq J$ and so $\wedge X \in J$ since J is a filter. Now we have $\wedge X \leq \vee Y$, since m_0 is an interpretation and so $\vee Y \in J$. As J is prime, we have $m_0(b) \in J$ for some $b \in Y$. So $Y \cap m_0^{-1}(J)$ is inhabited.

On the other hand, suppose x is a ideal element. We show that J_x is a completely prime filter. It is easy to see that it is a filter. To show it is completely prime, we use the concrete representation of elements in H_0 as conjunctive states. By Lemma 1, $\wedge X = \mathsf{c}\{X\}$. It suffices to show that if $X \subseteq x$ and $\mathsf{c}\{X\} \subseteq \mathsf{c}(\bigcup_{i \in I} u_i)$, where u_i are conjunctive states, then there exists some $Y \subseteq x$ and $i \in I$ such that $Y \in u_i$. For this it is enough to notice that whenever we can apply the rule

$$\frac{a_1, Z \quad \ldots \quad a_n, Z}{Z} \qquad \textit{provided} \ \ X \vdash a_1, \ldots, a_n \in T \quad X \subseteq Z$$

and $Z \subseteq x$ then there exists i such that $Z, a_i \subseteq x$. Indeed there exists i such that $a_i \in x$ because x is an ideal element and $X \subseteq x$. ⊠

Note that ideal elements need not exist for an arbitrary sequent structure. In particular, if we allow $\emptyset \vdash \emptyset$, then there is no way to obtain an ideal element. However, we have this basic result:

Theorem 12 (Completeness). *Every theory T on A has* enough *ideal elements:* $X \vdash_T Y$ iff *for all ideal elements x, the set $x \cap Y$ is inhabited whenever $X \subseteq x$.*

This theorem is an immediate consequence of Theorem 6 and Theorem 11 above.

It is worth noting a number of consequences of Theorem 12. First, notice that rule (C) is a form of the resolution rule, so that we get as a consequence completeness of resolution.

Secondly, as a special case of Theorem 12, we have $\emptyset \vdash \emptyset$ if and only if the sequent structure does not have any ideal element. This is precisely when the generated distributive lattice $\mathsf{L}(T)$ is *degenerated*, i.e., $0 = 1$.

Thirdly, from the proof of Theorem 12 we see that for any finite set $X \subseteq A$, there is an ideal element containing X if and only if $X \not\vdash_T \emptyset$.

Call $\wedge X$ a *semantical consequence* of $\wedge X_1, \ldots, \wedge X_n$ if for any ideal element x, $X \subseteq x$ implies $X_i \subseteq x$ for some i. We have the following completeness result, which is dual to Theorem 7.

Theorem 13. *The following are equivalent:*

1. $\wedge X$ *is a semantical consequence of* $\wedge X_1, \ldots, \wedge X_n$
2. $\wedge X \leq \wedge X_1 \vee \cdots \vee \wedge X_n$ *in* $\mathsf{Frm}(T)$
3. $X \vdash a_1, \ldots, a_n$ *for any choice* $a_1 \in X_1, \ldots, a_n \in X_n$

7 More categorical equivalence

Theorem 9 gives rise to a concrete description of the category **Spec** of spectral frames as sequent structures.

Let **Seq** be the category of sequent structures, where a map $f : A \to B$ is simply a map which preserves entailment: $X_1 \vdash X_2$ in A implies $f(X_1) \vdash f(X_2)$ in B.

If A, B are sequent structures, we define an *approximable relation* from A to B to be an interpretation $m : A \to \mathsf{Frm}(B)$ of A in the frame generated by B.

In view of Theorem 9, this can be directly described as a relation $\vdash$ between finite subsets of B and elements of A satisfying the following conditions:

- for any $x \in A$ the set of all $Y \subseteq B$ such that $Y \vdash x$ is a conjunctive state,
- if we have $x_1, \ldots, x_n \vdash_A u_1, \ldots, u_m$ and $Y \vdash x_i,\ 1 \leq i \leq n$ then there exists $Y_1, \ldots, Y_m$ such that $Y_j \vdash u_j,\ 1 \leq j \leq m$ and Y can be deduced from $Y_1, \ldots, Y_m$ using the rules (M) and (H).

By standard categorical construction (see for instance [13], Chapter VI, 5) we get that sequent structures with approximable maps form a category **RelSeq**.

Theorem 14. *The categories* **RelSeq**, **Spec** *are equivalent.*

Proof. This again follows from [13], Chapter VI, 5, Exercise 2). ⊠

8 Disjunctive states and hyperresolution

We apply now our results to the converse of a given relation on $\mathrm{Fin}(S)$ to get a proof of completness of hyperresolution, and an universal characterization of the lattices of disjunctive states.

It is clear that the hyperresolution rule (Section 4) is equivalent to the rules

$$\frac{Y}{X} \quad if\ Y \subseteq X$$
$$\frac{a_1, Z \quad \ldots \quad a_n, Z}{Z} \quad provided\ \ a_1, \ldots, a_n \vdash Y \in T \ \ Y \subseteq Z$$

(A simple combinatorial argument on permutation of rules show that we can even suppose the use of this last rule limited to the leaves of the derivation tree.) Let T_1 be the theory $\{Y \vdash X \ \mid \ X \vdash Y \in T\}$.

Theorem 15. *The complete lattice of all disjunctive states of a theory T is the frame generated by the theory T_1.*

In particular, there is a canonical correspondence between *points* of the frame of all disjunctive states and *co-elements* of a theory T.

It is clear that $\mathsf{L}(T_1)$ is the dual of $\mathsf{L}(T)$. By duality, it follows from our results that X is derived by hyperresolution from $X_1, \ldots, X_n$ iff

$$\wedge X \leq \wedge X_1 \vee \cdots \vee \wedge X_n$$

holds in $\mathsf{L}(T_1)$ iff

$$\vee X_1 \wedge \cdots \wedge \vee X_n \leq \vee X$$

holds in $\mathsf{L}(T)$ or equivalently, in $\mathsf{Frm}(T)$. Using Theorem 6 for the spectral frame $\mathsf{Frm}(T)$, this is true if and only if any point of $\mathsf{Frm}(T)$ containing $\vee X_1, \ldots, \vee X_n$ contains also $\vee X$, which means exactly that the clause X is a semantical consequence (Section 4) of the clauses $X_1, \ldots, X_n$.

Thus, we get in this way yet another derivation of the completeness of the hyperresolution rule, Theorem 7 (see [14, 25] as well). It is interesting to compare our argument to the proof in [2], which shows completness of hyperresolution by an abstract argument relying on Zorn's lemma. Our treatment has the advantage to structure the argument in two parts, one of which is purely algebraic and constructive, and the other is simply the general fact that spectral frames are spatial.

By soundness of the cut rule (C), which is nothing else than a form of the resolution rule, this gives an abstract, constructive proof that any resolution proof can be transformed into a hyperresolution proof. In particular, this shows the equivalence between $\vdash^*_{hr}$ and the "choice inference" $\dashrightarrow$, as stated in Theorem 7.

9 Example in algebra: spectrum of a ring

Let us give an example in algebra which illustrates some of the notions introduced here.

Let A be a commutative ring, and consider the theory T

- $\vdash 0$
- $x \vdash xy$
- $x, y \vdash x + y$
- $xy \vdash x, y$
- $1 \vdash$

Let $M(X)$ be the multiplicative monoid generated by X and $Idl(X)$ the ideal generated by X. We have the following direct description of $\vdash_T$.

Theorem 16. *$X \vdash_T Y$ if and only if $M(Y)$ meets $Idl(X)$.*

Proof. We prove first that this relation "$M(Y)$ meets (X)" is an entailment relation, which satisfies all the rules above. We analyse only the rule (C), the other rules being directly checked: assume that we have both $X \vdash Y, a$ and $a, X \vdash Y$. We reason in $A/Idl(X)$: by assumption we have $m_1, m_2 \in M(Y)$ and $a, x \in A$ and n such that $m_1 = ax$ and $m_2 a^n = 0$. This implies $m_1^n m_2 = 0$ and hence $0 \in M(Y)$ in $A/Idl(X)$. Hence $X \vdash Y$ as required.

It is direct that this entailment relation satisfies all the rules above.

Conversely, if the $M(Y)$ meets $Idl(X)$, we can derive $X \vdash Y$ using only the given axioms. Indeed, the first third rules show that $X \vdash y$ whenever $y \in Idl(X)$ while the two last rules show $y \vdash Y$ whenever $y \in M(Y)$. ⊠

In the particular case where A is a ring of polynomials, we have a proof of the formal Nullstellensatz theorem presented in [12]: the following items

- $x_1, \ldots, x_n \vdash y$ is a consequence of the above axioms
- y belongs to the radical of the ideal generated by $x_1, \ldots, x_n$
- $\{y\}$ can be derived from $\{x_1\}, \ldots, \{x_n\}$ by hyperresolution

are equivalent.

An ideal element of this entailment relation is a prime ideal of A. Furthermore, if I is a radical ideal of A, then the set of finite subsets whose product is in I is a disjunctive state U_I. Conversely, if U is a disjunctive state, and I is the set of elements x such that $\{x\} \in U$ then I is a radical ideal such that $U = U_I$.

10 Application: Representation of stably compact frames

We apply our results to obtain a concrete representation of stably compact frames. We show how to use this notion to give a concrete description of the patch construction introduced by Escardo [7].

Let $<$ be a transitive and dense relation on the set A of tokens – *dense* in the sense that if $x < y$ then there exists $z \in A$ such that $x < z < y$. Let D_a stand for the set $\{b \in A \mid b < a\}$.

Let T be a theory on A equiped with a transitive and dense relation $<$. Such a theory T is said to satisfies the *interpolation property* if whenever $a_1, \dots, a_n \vdash b_1, \dots, b_m \in T$ and $c_i < a_i$ for $i = 1 \dots n$, there exist finite sets $Y_j \subseteq D_{b_j}$ such that

$$c_1, \dots, c_n \vdash_T Y_1, \dots, Y_m$$

Recall that $Y_1, \dots, Y_m$ stands for $Y_1 \cup \dots \cup Y_m$.

We say that an interpretation $m : A \to H$ of T in a frame H is *continuous* iff $m(a) = \bigvee\{m(b) \mid b < a\}$.

Theorem 17. *We suppose that T is a theory on A such that $a \vdash_T b$ if $a < b$ and T satisfies the interpolation property. If $H, m : A \to H$ is an universal continuous interpretation the frame H is* stably compact *and is a retract of* $\mathsf{Frm}(T)$.

Proof. We know by theorem 5 that $\mathsf{Frm}(T)$, $m_0 : A \to \mathsf{Frm}(T)$ is a spectral frame. By universality we have a unique frame map $f : \mathsf{Frm}(T) \to H$ such that $f \circ m_0 = m$. We build a frame map $g : H \to \mathsf{Frm}(T)$ such that $f \circ g = id_H$, so that H is a *retract* of $\mathsf{Frm}(T)$. For this, we give a continuous interpretation $i : A \to \mathsf{Frm}(T)$

$$i(a) = \bigvee\{m_0(b) \mid b < a\}$$

This is an interpretation of T by the interpolation property. It is continuous since $<$ is dense. Hence, by universality of H we have a map $g : H \to \mathsf{Frm}(T)$ such that $g(m(a)) = i(a) = \bigvee\{m_0(b) \mid b < a\}$. Notice that

$$f(g(m(a))) = \bigvee\{f(m_0(b)) \mid b < a\} = \bigvee\{m(b) \mid b < a\} = m(a)$$

and hence $f \circ g = id_H$ by universality. It follows that H, retract of a spectral frame, is stably compact (cf. theorem VII 4.6 of [9]). ⊠

As a typical example, the interval $[0, 1]$ can be described by the set of token L_r, $r \in Q$ with the relation $L_r < L_s$ iff $r < s$ and the theory

- $L_r \vdash$ if $r \leq 0$
- $\vdash L_r$ if $1 < r$
- $L_r \vdash L_s$ if $r < s$

We use this construction to describe the patch topology [7]. The set of token S is now $A \cup \{a' \mid a \in A\}$ where a' are new token. We extend the theory T with the axioms

- $\vdash a', b$
- $a, b' \vdash$

if $a < b$ and we call *patch*(T) the resulting theory.

We simply state the representation theorem.

Theorem 18. *The frame $m : S \to H$ universal as an interpretation of patch(T) satisfying*

$$m(a) = \bigvee\{m(b) \mid b < a\} \qquad m(a') = \bigvee\{m(b') \mid a < b\}$$

is compact regular. Furthermore, it is the patch reflection [7] of the universal stably compact continuous interpretation of T.

As a typical example, the patch reflection of the previous topology on $[0, 1]$ will be obtained by adding the new token $L'_r = U_r$ for $r \in Q$ and the axioms

- $\vdash U_r, L_s$
- $L_r, U_s \vdash$

if $r < s$. We get then a description of $[0, 1]$ with its usual topology.

11 Concluding remarks

Sequent structures are the skeletons of propositional theories. A propositional theory can be reduced to a sequent structure by translating an entailment instance $\varphi_1 \vee \varphi_2 \vdash \psi_1 \wedge \psi_2$ to simpler ones $\varphi_i \vdash \psi_j$ $(i, j \in \{1, 2\})$ repeatedly until only $\wedge$ appear on the left, and only $\vee$ appear on the right (distributivity is used in this process). The remaining $\wedge$'s and $\vee$'s can then be removed by virtue of sequents. Of course this process can be reversed; but we believe that working at the sequent level can in many cases avoid tedious syntactic details.

We end by repeating the hope given in the introduction that this paper be a first step in exploring completeness of various logical systems by means of canonical embedding to locales. It should be interesting to develop richer tools for this purposes, in order to handle additional logical operators. The well-known Henkin construction for instance, has been investigated in this setting [16] for linear logic.

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